38.1 Dirac Particle in an Electromagnetic Potential

Since it costs us very little, we might as well see how electromagnetic potentials fit into the Dirac equation. We can introduce electromagnetic potentials in exactly the same way we introduced them into the Klein-Gordon equation. After all, the two approaches stem from the same Hamiltonian (33.29). The Dirac equation with E&M potentials is written

\[
(H - q\phi)\Psi = c \left( \vec{p} - q\vec{A} \right) \cdot \vec{\alpha}\Psi + mc^2\beta\Psi,
\]

where \( H = i\hbar \frac{\partial}{\partial t} \) and \( \vec{p} = -i\hbar \vec{\nabla} \). In differential form, this is

\[
(i\hbar \frac{\partial}{\partial t} - q\phi)\Psi = c \left( -i\hbar\vec{\nabla} - q\vec{A} \right) \cdot \vec{\alpha}\Psi + mc^2\beta\Psi
\]  

Again, we may write this equation while showing the four components of \( \Psi \) explicitly:

\[
\begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
\Psi_3 \\
\Psi_4
\end{pmatrix}
= c \begin{pmatrix}
-\psi_1 - ic(\vec{\alpha}\nabla - q\vec{A}) \\
\psi_1 c \vec{\alpha} - q\vec{A} \\
\psi_2 c \vec{\alpha} + q\vec{A} \\
\psi_2 - ic(\vec{\alpha}\nabla - q\vec{A}) \\
\psi_3 c \vec{\alpha} + q\vec{A} \\
\psi_3 + ic(\vec{\alpha}\nabla - q\vec{A}) \\
\psi_4 c \vec{\alpha} - q\vec{A} \\
\psi_4 + ic(\vec{\alpha}\nabla - q\vec{A})
\end{pmatrix}
\]

The probability density is still

\[
\rho = \Psi^\dagger\Psi
\]

even in the presence of EM influences. The probability current density, which obeys

\[
\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0
\]

is also unchanged:

\[
\vec{J} = \Psi^\dagger c\vec{\alpha}\Psi
\]

Proof:

\[
\frac{\partial \rho}{\partial t} = \Psi^\dagger \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^\dagger}{\partial t} \Psi = \Psi^\dagger \frac{1}{i\hbar} \left[ c \left( -i\hbar\vec{\nabla} - q\vec{A} \right) \cdot \vec{\alpha}\Psi + q\phi\Psi + mc^2\beta\Psi \right] + \frac{1}{i\hbar} \left[ c \left( i\hbar\vec{\nabla} - q\vec{A} \right) \cdot \vec{\alpha}\Psi^\dagger + q\phi\Psi^\dagger + mc^2\beta\Psi^\dagger \right] \Psi
\]

\[
= -\Psi^\dagger c\vec{\alpha} \cdot \nabla \Psi - c \left( \nabla \Psi^\dagger \right) \cdot \vec{\alpha}\Psi = -\Psi^\dagger c\vec{\alpha} \cdot \nabla \Psi - \left[ c\nabla \cdot (\Psi^\dagger \vec{\alpha}\Psi) - \Psi^\dagger c\vec{\alpha} \cdot \nabla \Psi \right] = -\nabla \cdot (\Psi^\dagger c\vec{\alpha} \Psi)
\]

(38.7)
38.2 The Schrödinger Equation with a Spinor

In chapters 4 and 5 of our textbook, we appended, in an *ad hoc* fashion, a spinor to our usual Schrödinger wave function. See, for example, Eq. (5.23), where we wrote the overall wave function as $\psi(\vec{r}) \chi(\vec{s})$. The spinor $\chi(\vec{s}) = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ is a state acted on by operators such as

$$S = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{x} + \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \hat{y} + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{z} = \frac{\hbar}{2} \sigma_x \hat{x} + \frac{\hbar}{2} \sigma_y \hat{y} + \frac{\hbar}{2} \sigma_z \hat{z} = \frac{\hbar}{2} \hat{\sigma}$$

The Schrödinger equation does not tell us to do this, but the Dirac equation does, as we shall now demonstrate. We will take the non-relativistic limit of the Dirac equation, and recover the two-dimensional spinner form that we previously encountered.

As was noted above, the Dirac equation (38.3) can be written as four first-order coupled differential equations. We can also organize the 4-vector equation into two coupled 2-vector equations:

$$\begin{pmatrix} i \hbar \frac{\partial}{\partial t} - q \phi \\ \Psi_i \end{pmatrix} = c \begin{pmatrix} -i \hbar \frac{\partial}{\partial x} - qA \end{pmatrix} \begin{pmatrix} \Psi_i \\ \Psi_4 \end{pmatrix} + c \begin{pmatrix} -i \hbar \frac{\partial}{\partial y} - qA \end{pmatrix} \begin{pmatrix} -i \Psi_4 \\ i \Psi_3 \end{pmatrix} + c \begin{pmatrix} -i \hbar \frac{\partial}{\partial z} - qA \end{pmatrix} \begin{pmatrix} \Psi_3 \\ -i \Psi_4 \end{pmatrix} + mc^2 \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

These can be written more compactly as

$$\begin{pmatrix} i \hbar \frac{\partial}{\partial t} - q \phi \\ \Psi_i \end{pmatrix} = c \begin{pmatrix} -i \hbar \nabla - q\vec{A} \end{pmatrix} \cdot \sigma \begin{pmatrix} \Psi_i \\ \Psi_4 \end{pmatrix} + mc^2 \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

(38.9)

where

$$\sigma = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \hat{y} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{z}$$

(38.10)

When solving for states in an atom, we often considered stationary states. If $\vec{A}$ and $\phi$ are time-independent, then we may write (as usual):

$$\Psi_1(t, \vec{r}) = \psi_1(\vec{r}) e^{-i E_1 \hbar}, \quad \Psi_2(t, \vec{r}) = \psi_2(\vec{r}) e^{-i E_2 \hbar}, \quad \text{etc.}$$

(38.10)

When these are substituted into (38.9), we arrive at

2
\[
\begin{align*}
(E - q\phi)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= c(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} + mc^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
(E - q\phi)\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} &= c(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - mc^2 \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}
\end{align*}
\] (38.11)

We write the energy as
\[
E = \tilde{E} + mc^2,
\] (38.12)
where \( \tilde{E} \) is all energy besides rest energy (i.e. kinetic plus potential), which gives
\[
\begin{align*}
\tilde{E} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= c(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} + q\phi \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
(\tilde{E} - q\phi + 2mc^2) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} &= c(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\end{align*}
\] (38.13)

The term \( mc^2 \) neatly cancels out of the first equation.

So far, everything is exact; (38.13) is equivalent to the Dirac equation. Now for the nonrelativistic limit: we will suppose that \( \tilde{E} \) and \( q\phi \) are small compared to \( mc^2 \). Our second equation (38.13) then reduces to
\[
\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \frac{1}{2mc}(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\] (38.14)
which we substitute into the first equation to get
\[
\begin{align*}
\tilde{E} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \frac{1}{2m}(-i\hbar \nabla - q\vec{A}) \cdot \sigma(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + q\phi \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&= \frac{1}{4mc}(\nabla \cdot \vec{A}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + q\phi \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\end{align*}
\] (38.15)

Notice that \( \psi_1 \) and \( \psi_2 \) are now decoupled from \( \psi_3 \) and \( \psi_4 \). Recall that these wave parameters are associated with electron spin up and down as well as positron spin up and down, respectively. Since we are shooting for a non-relativistic regime, it feels good to be rid of the positrons states. However, keep in mind that we accomplished this by substitution, not merely by setting the positron states to zero. If we would have set the positron states \( \psi_3 \) and \( \psi_4 \) to zero, we would have forced \( \psi_1 \) and \( \psi_2 \) to be zero as well.

Therefore, as we reach the Schrödinger equation in what follows, we may conclude that it is the existence of positron amplitude in the Dirac equation that leads to the wave behavior of the electron as described by the Schrödinger equation.

Moving onward now, it is left as an exercise to show that
\[
(-i\hbar \nabla - q\vec{A}) \cdot \sigma(-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (-i\hbar \nabla - q\vec{A}) \cdot \sigma \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + q\hbar \sigma \cdot (\nabla \times \vec{A}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\] (38.15)
Since $\vec{B} = \nabla \times \vec{A}$ and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, we get

$$E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2m} \left( -i\hbar \nabla - q\vec{A} \right)^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \frac{q}{m} \vec{S} \cdot \vec{B} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + q\phi \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{38.16}$$

This is the time-independent Schrödinger equation with a spinor attached just like we have seen, for we may write

$$\begin{pmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{pmatrix} \equiv \begin{pmatrix} \chi_1 \psi(\vec{r}) \\ \chi_2 \psi(\vec{r}) \end{pmatrix} = \psi(\vec{r}) \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{38.17}$$

The $\vec{A}$ inside of $(-i\hbar \nabla - q\vec{A})^2$ takes care of the influence of a magnetic field on orbital angular momentum (as well as providing $-\hbar^2 \nabla^2$). We will not bother putting the spin-orbit coupling into the more familiar form (section 6.3.1)).

On the other hand, we have seen the perturbative Hamiltonian $H_{\text{spin-mag}} = \frac{q}{m} \vec{S} \cdot \vec{B}$ before, Eqs. (4.157) and (6.71), which includes the mysterious gyromagnetic ratio 2!!!! (See footnote p. 178.) The Dirac equation gets it right while classical physics fails in this regard.