Driven simple harmonic oscillator solution

The next two pages compare solutions with real functions (cos, sin), and complex functions (exp)

The goal is to get the familiar amplitude and phase vs frequency:

\[
Q = \frac{\omega_0}{\gamma}
\]

\[
\varphi = -\arctan\left(\frac{\gamma \omega}{\omega_0^2 - \omega^2}\right)
\]
Driven simple harmonic oscillator using **real functions**:

Lorentz model of charge on spring (constant \(k\)), driven by an oscillating E-field.

Use one dimension, position called \(r\) instead of \(x\).

\[
F_{\text{spring}} = -kr = -m\omega_o^2 r \quad F_{\text{damping}} = -\gamma mv = -\gamma m\dot{r} \quad (\dot{r} \text{ means one time derivative, } \ddot{r} \text{ second derivative})
\]

\[
F_{\text{driving}} = qE \quad \text{so} \quad \sum F = m\ddot{r} \text{ becomes } \dddot{r} + \gamma \dot{r} + \omega_o^2 r = \frac{q}{m} E_o \cos(\omega t)
\]

Solution \(r\) will be of the form: \(r = r_o \cos(\omega t + \phi)\)

\[
\cos(a + b) = \cos a \cos b - \sin a \sin b
\]

so \(r = r_o \cos(\omega t + \phi) = r_o (\cos \phi \cos \omega t - \sin \phi \sin \omega t) = r_1 \cos \omega t + r_2 \sin \omega t\)

Then the oscillation amplitude (magnitude) is \(r_o = \left( r_1^2 + r_2^2 \right)^{\frac{1}{2}}\), and \(\phi = \arctan \left( -\frac{r_2}{r_1} \right)\)

\[
\dddot{r} + \gamma \dot{r} + \omega_o^2 r = \frac{q}{m} E_o \cos(\omega t)
\]

\[
-\omega_o^2 (r_1 \cos \omega t + r_2 \sin \omega t) + \gamma \omega (-r_1 \sin \omega t + r_2 \cos \omega t) + \omega_o^2 (r_1 \cos \omega t + r_2 \sin \omega t) = \frac{q}{m} E_o \cos(\omega t)
\]

Now the coefficients in front of the \(\cos\) and \(\sin\) terms must be equal on both sides of the equation:

\[
\left[ (\omega_o^2 - \omega^2) r_1 + \gamma \omega r_2 \right] \cos \omega t = \frac{q}{m} E_o \cos(\omega t)
\]

\[
\left[ (\omega_o^2 - \omega^2) r_1 - \gamma \omega r_2 \right] \sin \omega t = 0
\]

Solve these two equations for \(r_1\) and \(r_2\):

\[
\frac{r_2}{r_1} = \frac{\gamma \omega}{(\omega_o^2 - \omega^2)}
\]

so \(r_1 \left[ (\omega_o^2 - \omega^2) + \gamma \omega \frac{\gamma \omega}{(\omega_o^2 - \omega^2)} \right] = \frac{q}{m} E_o\)

\[
r_1 = \frac{q}{m} E_o \left[ \frac{(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2} \right]
\]

and \(r_2 = \frac{q}{m} E_o \left[ \frac{\gamma \omega}{(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2} \right]\)

\[
\phi = \arctan \left( -\frac{r_2}{r_1} \right) = \arctan \left( -\frac{\gamma \omega}{(\omega_o^2 - \omega^2)} \right)
\]
Complex solution. See how much simpler it is:

\[ E = (E_o e^{i\omega t}) \]. The physical \( E \) is \( E = \text{Re}(E_o e^{i\omega t}) \), but we will take \( \text{Re} \) of everything at the end.

Complex \( r = r_o e^{i\omega t} \), where \( r_o = (r_1 + ir_2) = |r_o| e^{i\phi} \). \( |r_o| = \left( r_1^2 + r_2^2 \right)^{1/2} \) \( \phi = \arctan \left( \frac{r_2}{r_1} \right) \).

The physical \( r \) is \( \text{Re}(r_o e^{i\omega t}) = |r_o| \cos(\omega t + \phi) \), but we will take \( \text{Re} \) later.

Each derivative in time brings down factor \( i\omega \):

\[ \ddot{r} + \gamma \dot{r} + \omega_o^2 r = \frac{q}{m} E_o e^{i\omega t} \] becomes:

\[ -\omega_o^2 r_o e^{i\omega t} + i\gamma \omega r_o e^{i\omega t} + \omega_o^2 r_o e^{i\omega t} = \frac{q}{m} E_o e^{i\omega t} \]

\[ r_o = \frac{q}{m} E_o \frac{1}{\omega_o^2 - \omega^2 + i\gamma \omega} = r_1 + ir_2 \] This is beautifully compact!

If you split \( r_o \) into its real and imaginary parts to find \( r_1 \) and \( r_2 \), they are the same (within a sign) as on the previous page, as expected since the phasor picture unites the two descriptions.

\[ \varphi = \arctan \left( \frac{r_2}{r_1} \right) = \arctan \left( \frac{-\gamma \omega}{\left( \omega_o^2 - \omega^2 \right)} \right) \]

The ratio of the imaginary part to the real part determines the phase of oscillation at each frequency. Complex numbers keep track of phase.

\[ |r_o| = \left( r_1^2 + r_2^2 \right)^{1/2} = \left[ r_o r_o^* \right]^{1/2} \]

\[ = \left[ \frac{q}{m} E_o \frac{1}{\omega_o^2 - \omega^2 + i\gamma \omega} \frac{q}{m} E_o \frac{1}{\omega_o^2 - \omega^2 + i\gamma \omega} \right]^{1/2} \]

\[ |r_o| = \frac{q}{m} E_o \left[ \frac{1}{\left( \omega_o^2 - \omega^2 \right)^2 + (\gamma \omega)^2} \right]^{1/2} \]