A. A Bose-Einstein condensation in a gas of sodium atoms takes place at about 2 \( \mu \text{K} \).

(i) What is the thermal wavelength at this temperature?

(ii) Find the density from \( \frac{N}{V} = 1 \text{ particle/} \lambda^3 \). Find the pressure that this gas would exert on its trap, from the ideal gas law.

(iii) At that density, to what temperature would you have to cool the atoms in order to get 99% of the atoms into the condensate? Use
\[
N_{\text{ground}}(T) = N \left( 1 - \left( \frac{T}{T_c} \right)^{3/2} \right) \quad \text{for } T < T_c.
\]

B. Suppose I have a system consisting of three single-particle-states 1, 2, 3, with energies \( E_1, E_2 \) and \( E_3 \), at temperature \( T \). Imagine for example the first three levels of a finite square well. But in this system, I have two particles which might occupy the single-particle-states. Suppose that the energies of the three states are \( E_1 = 0 \), \( E_2 = \Delta \) and \( E_3 = 3 \Delta \).

(i) Make diagrams showing all the possible arrangements (two-particle-states) assuming the particles are identical fermions (for example electrons) and label each arrangement with its energy. You should get 3 two-particle-states. (Ignore any spin freedom, so only one particle per single-particle-state). One arrangement is shown at the right. Let’s call the arrangements (2-particle states) a, b, c.

(ii) Write the relative probabilities \( P_a/P_a, P_c/P_a \), from Boltzmann’s theorem, and, knowing that \( P_a + P_b + P_c = 1 \), solve for \( P_a \) first, then \( P_b \) and \( P_c \). Find the average energy \( \langle E \rangle = \sum E_i P_i \).

(iii) Find \( N_1 \) vs \( T \) (this is the average number \( \langle N_1 \rangle \) in the ground state, the lowest single-particle-state, 1.) To do this, find the number of particles in the single-particle-state 1, for each two-particle state (arrangement). Label these \( N_{1i} \). Then \( \langle N_1 \rangle = \sum N_{1i} P_i \).

(iv) Make diagrams showing all the possible arrangements (two-particle-states) assuming the particles are identical bosons (like \( ^4 \text{He} \) atoms), and label each with its energy. You should get 6 two-particle-states. One arrangement is shown at the right. Let’s call the arrangements a, b, c, etc.

(v) Write the relative probabilities \( P_a/P_a, P_c/P_a \), etc for all the states from Boltzmann’s theorem, and, knowing that \( P_a + P_b + P_c + \ldots = 1 \), solve for all six \( P \)’s. Find the average energy \( \langle E \rangle = \sum E_i P_i \), and \( \langle N_1 \rangle = \sum N_{1i} P_i \).

(vi) Consider three temperatures, \( kT = 0 \), \( kT = \Delta \), and \( kT = 100 \Delta \). Evaluate \( \langle N_1 \rangle \) at these temperature (just a number) from your results for fermions and bosons. Which is greater? Does it match what we know about these statistics? At what \( T \)’s does the kind of particle make the biggest difference?

(vii) For the fermions, it must also be true that \( \langle N_1 \rangle = f_{FB} \left( E_i \right) = \frac{1}{e^{\alpha E_i / kT} + 1} = \frac{1}{e^{\frac{(E_i - E_f)}{kT}} + 1} \). Find the normalization constant \( \alpha \) and \( E_f \) for the temperatures \( kT = 0 \) and \( kT = \Delta \). Knowing that \( E_f \) is the energy that roughly divides occupied from unoccupied states at low \( T \), does your value for \( E_f \) make sense compared to \( \Delta \)?

(viii) For the bosons, it must also be true that \( \langle N_1 \rangle = f_{BE} \left( E_i \right) = \frac{1}{e^{\alpha E_i / kT} - 1} \). Find \( \alpha \) for the temperatures \( kT = 0 \) and \( kT = \Delta \).