1. (14 points) Expand the wave below as a Fourier series. Note that the minimum and maximum values of this function are ±0.5. Also note that, if \( n \) is an integer, then
\[
\int_0^1 x \sin(n \pi x) \,dx = \frac{(-1)^{n+1}}{n \pi}.
\]

2. (13 points) In this problem we are going to derive the formulas for the constants in finite Fourier sine and cosine series from the Euler equations for the Fourier coefficients in a Fourier series.

(a) Assume I have a function, \( f(x) \), which is only defined from \( x = 0 \) to \( x = L \). I want to write it as a sum of sines of the form
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin(n \pi x / L).
\]
To do this, we will imagine another function \( g(x) \) which is periodic and is equal to \( f(x) \) from \( x = 0 \) to \( L \). I can write \( g(x) \) as
\[
g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n k_0 x) + b_n \sin(n k_0 x).
\]
But we don’t want to imagine \( g(x) \) to be just any periodic function which happens to equal \( f(x) \) from \( x = 0 \) to \( L \). For starters, when we write it as a Fourier series, we only want sines in our sums - we want the cosine terms to go to zero. What does that tell us about the periodic function \( g(x) \)?

(b) If we want the Fourier series of \( g(x) \) to look like our sum representation of \( f(x) \), what does \( k_0 \) have to be, and what will be the period of our periodic function \( \lambda_0 \)?
(c) Show that with the above properties I can write \( g(x) \) in the form

\[
g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x / L)
\]

and find a formula for \( b_n \) which only involves \( f(x) \) (not \( g(x) \)), the constants \( n, \pi, \) and \( L \) (not \( \lambda_0 \) or \( k_0 \)), the variable \( x \), and integration from \( x = 0 \) to \( L \). (Hint - what happens when I integrate an even function from \(-a\) to \( a\), and what do I get when I multiply two odd functions together?)

(d) Now assume that I want to write \( f(x) \) in the form

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x / L).
\]

Using steps similar to the ones you did above, find a formula to find what \( a_n \) is equal to.

3. (13 points) Imagine that I have a guitar. I grab one of the strings right in the middle, and I pull it out a distance \( h \).

(a) Write this shape as a sum of harmonics on the string, of the form

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right).
\]

Note that

\[
\int x \sin(x) \, dx = -x \cos(x) + \sin(x).
\]

(b) Check your answer by plotting this on a computer. Only sum up to \( n = 50 \), and let \( h = L = 1 \).

(c) Each harmonic in the sum oscillates sinusoidally at a fixed frequency \( \omega_n \). So to find how the string oscillates, we just multiply each term in our series by \( \cos(\omega_n t) \) to get an equation that describes the motion of the string in time (note that if the string weren’t motionless at time \( t = 0 \), there could be a phase factor as well, which would change the problem a bit). Plot what the string will look like at times \( t = 0, 0.05, 0.10 \) and \( 0.15 \). Let \( h = 1 \) and \( \omega_n = 2\pi n \), and only sum up to \( n = 50 \).