You’ve got a bunch of meatballs that you’ve just defrosted, and you want to boil them in a stew. You’ve read that you need to cook meatballs until the centers get to 74°C. You also looked around on the internet and found that the specific heat of meatball stuff is typically around 3520 J/Kg°C, the density of meatball is around 1000 kg/m³, and the heat conductivity of meatball stuff is typically around 0.4 W/m°C. Yes, I really looked all that up.

1. (10 points) Let’s first define the problem we’re going to solve.
   
   (a) Write down the equation we’ll need to solve to find the temperature $T$ inside our meatball at all times. Let’s keep it general for now - write it using the Laplace operator. And write it in terms of $\rho$, $k$, and $c$ (the density, heat conductivity, and specific heat of meatball gunk).
   
   (b) What coordinate system naturally fits this problem? Cartesian? Cylindrical? Spherical? (Hint - meatballs are not shaped like cubes or cylinders.)
   
   (c) Now write down the equation in that coordinate system. Then use the symmetry of the problem to throw out derivatives with respect to two of the independent variables such that you are left with a PDE that only involves time and one spatial coordinate.
   
   (d) Let’s assume that we’re going to drop the meatball into boiling water, and let’s assume that the water is moving enough that we never develop a significant “boundary layer” around the meatball, so that we can assume that the temperature at the edge of the meatball is always 100°C. Write an equation that represents this boundary condition. Assume that the radius of the meatball is $a$. Since we only have one boundary, the other boundary condition will be “It doesn’t blow up anywhere inside the meatball.”

2. (15 points) Now let’s find the general solution to the problem.

   (a) Things will be nicer if we homogenize the boundary condition. Transform your equation for $T(r,t)$ into an equation for $W(r,t)$ which has homogeneous boundary conditions. Write down the transformation, the new PDE, and the new boundary conditions.
   
   (b) Now let’s do separation of variables - let’s define $W(r,t) = f(t)g(r)$ and plug that into your PDE. Then define your separation constant such that you get

   $$f_t = -\frac{\alpha^2 k}{\rho c} f$$

   for your time equation. What do you get for your $r$ equation?
   
   (c) What is the solution to the $t$ equation?
   
   (d) The $r$ equation isn’t like any we’ve solved before. But I asked Mathematica, and it told me that the solution to the problem

   $$G''(x) + \frac{2}{x} G'(x) + \beta G = 0$$

   is

   $$G(x) = C_1 \frac{1}{x} e^{-\sqrt{-\beta}x} + C_2 \frac{1}{x} e^{\sqrt{-\beta}x}.$$ 

   So, Mathematica did most of the work for us on this part. All you have to do is recognize that the square root of minus one is $i$, write this solution in a more sensible manner (involving sines and cosines), and then use it to find the solution to the $r$ equation. (Hint - your answer should involve two arbitrary constants, and each term should have a sine or a cosine of $\alpha r$ in them.)
(e) If you look at your answer to the last part, you should notice that one of the two terms blows up at \( r = 0 \). Apply the “It doesn’t blow up” boundary condition to find a simpler version of \( g(r) \). (By the way, in addition to not blowing up everywhere, it’s also important that the derivative with respect to \( r \) go to zero at \( r = 0 \). Otherwise, a finite power will be flowing in or out of the infinitesimal piece of meatball at \( r = 0 \). If you want to, you can plot your solution, and you will see that this is true. See - math is like magic!)

(f) Now apply the other boundary condition to find what values \( \alpha \) can have.

(g) Write down the most general solution for \( W(r, t) \) that satisfies the boundary conditions. Remember to include limits on your sum.

(h) Transform this back to find the general solution to \( T(r, t) \) that satisfies the boundary conditions.

3. (15 points) Next step, apply the initial conditions - which are that the meatball is initially at a temperature of 0°C everywhere.

(a) To apply the initial conditions, we need an orthogonality integral. Notice that if you choose the right kernel, this looks just like the orthogonality integral you got for the sine transform. What kernel should we use?

(b) Apply the initial condition and use the orthogonality integral to find the final solution to our problem. Note that

\[
\int x \sin(x) = \sin(x) - x \cos(x)
\]

I’m not going to require that you explicitly verify that your solution is indeed a solution to the PDE and that it satisfies the BC. But I suggest that you do.

(c) Plug in the specific heat, etc., for meatballs, assume that the meatball has a radius of 1 cm, and plot your solution at time \( t = 0.1, 10, \) and 100 seconds to verify that it satisfies your initial and boundary conditions. Only sum up to \( n = 100 \). (Note that if you plot at really early times, there are a lot of wiggles and the temperature at \( r = 0 \) is relatively high. This is because you cut your sum off at \( n = 100 \). The high \( n \) terms matter a lot at short times, but they die off quickly.) Attach a copy of your plot.

(d) Plot \( T(0, t) \) from time \( t = 0.1 \) to \( t = 200 \) seconds. From the plot, estimate about how many minutes it takes to cook a meatball.