1. (14 points) If \( n \) is a non-negative integer, you can generate the Legendre polynomials using the Rodrigues’ formula:

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

(a) Use this formula to find and plot the Legendre polynomials \( P_0(x), P_1(x), P_2(x), \) and \( P_3(x) \) from \( x = -1 \) to 1.

(b) What appears to be the trend for values of \( P_n(x) \) at \( x = \pm 1 \)?

(c) If \( n \) is odd, what does it appear that \( P_n(x) \) does at \( x = 0 \)?

It turns out that these patterns hold - so remember them, they can come in handy. Also note that if \( n \) is even, \( P_n \) will only have even powers of \( n \) and if \( n \) is odd, \( P_n \) will only have odd powers of \( x \), and \( n \) will be the highest power of \( x \) in \( P_n \).

2. (6 points) The Legendre functions of the second kind don’t show up nearly as often. But let’s take a quick look at them. The first few functions are

\[
Q_0(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)
\]

\[
Q_1(x) = \frac{x}{2} \ln \left( \frac{1+x}{1-x} \right) - 1
\]

\[
Q_2(x) = \frac{3x^2 - 1}{4} \ln \left( \frac{1+x}{1-x} \right) - \frac{3x}{2}
\]

(a) Plot \( Q_0(x), Q_1(x), \) and \( Q_2(x) \) from \( x = -1 \) to 1.

(b) What appears to be the trend for the values that \( Q_n(x) \) take at \( x = \pm 1 \)?

(c) What appears to happen to \( Q_n(x) \) for even \( n \) at \( x = 0 \)?

3. (10 points) OK, I don’t have a story for this problem. Consider the following ODE/BC problem to find \( Z(r) \) on the interval \( 0 \leq r \leq 1 \).

\[
(1-r^2)Z_{rr} - 2rZ_r + 6Z = 0
\]

\[
Z(0) = 1
\]

\[
Z(1) = \text{finite}
\]

(a) Find the general solution to this equation.

(b) Apply the boundary conditions to find \( Z(r) \).

(c) Verify that your solution matches the BC and satisfies the ODE.

4. (10 points) Again, no story. Solve this problem to find \( y(x) \) on the interval \( 0 \leq r < 1 \).

\[
(1-x^2)y'' - 2xy' - \frac{y}{x+1} \left( \frac{11 - 12x^2}{x - 1} \right) = 0
\]

\[
y(0) = 2
\]

\[
y(1) = \text{finite}
\]