1. (18 points) Imagine that I have a cookie sheet which is $W$ wide and $L$ long. I set it down upside down, so the bottom of the sheet is not touching the table. The bottom of the sheet is free to vibrate according to the wave equation

$$u_{tt} = c^2 \nabla^2 u.$$  

The side walls of the cookie sheet are stiff, preventing the edges of the bottom of the sheet from moving. This gives us the boundary conditions

$$u(0, y, t) = u(W, y, t) = u(x, 0, t) = u(x, L, t) = 0.$$  

(a) What is $\nabla^2$ in a 2D Cartesian coordinate system?

(b) Plug the 2D Cartesian coordinate representation of $\nabla^2$ into the PDE above. Then assume there are solutions of the form $u(x, y, t) = a(x)b(y)f(t)$. Plug this into the PDE and then divide by $abf$. You should then have an equation, one side of which explicitly only depends on time. But it can’t depend on time either, because the other side doesn’t depend on time. So set each side of the equation equal to the constant $-\alpha^2$, and separate the PDE into an ODE for $f$ and a PDE for $a$ and $b$. Find the general form of the solution to the $f$ ODE.

(c) Now look at your PDE for $a$ and $b$. Move the term involving $b$ to the other side of the equation. Now one side of the equation explicitly only depends on $x$, and the other side explicitly only depends on $y$. So that means that both sides must equal a constant. Set both sides equal to $-\beta^2$, and separate the PDE into two ODEs. Then find the general solutions to them.

(d) Find the general solution to the problem by multiplying your general solutions to $a$, $b$, and $f$, and then summing over all possible values of $\alpha$ and $\beta$ (i.e., it will be a double sum).

(e) Now apply your boundary conditions to get a general solution for $u(x, y, t)$ which fit the boundary conditions.

(f) What is the angular frequency of the lowest frequency mode of the cookie sheet?

2. (18 points) You strike your cookie sheet with a wooden spoon, giving it a set of initial conditions approximating

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = \delta(x - W/2)\delta(y - L/2).$$

(a) Find $u(x, y, t)$ for these initial conditions.

(b) Let $c = W = 1$ and $L = 2$ and plot your solution at times $t = 0, 0.25, 0.5, 0.75, 1, \text{ and } 1.25$. Note - the command in Mathematica to do a plot of a 2D surface (where the height of the surface is a function of two coordinates) is Plot3D. Oh, and for the sake of making this possible to do, only take each sum out to the 50th term. And you can make your plots look nicer if you use the options

$$\text{BoxRatios} \rightarrow \{1, 2, 0.5\}, \quad \text{PlotRange} \rightarrow \{-2, 2\}$$

3. (4 points) Show that any solution to the 3D wave equation $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$ which has no dependence on $z$ (in other words, $u(x, y, z, t)$ don’t actually have any $z$s in it), is also a solution to the 2D wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$. 