1. (6 points) Use the ratio test, integral test, and/or alternating series test find the region of convergence for the following series. State whether they are absolutely convergent or conditionally convergent in that region, or if they are divergent everywhere. (Note: the ratio test fails along the edge of the region for this assignment don’t worry whether the region of convergence has $a \geq 0$ or $\leq 0$. > or <, but realize that in some problem you work in the future it may matter, and you will have to be more careful.)

(a) $f(z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$

(b) $f(z) = \sum_{n=1}^{\infty} e^{nz}$

(c) $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$

2. (4 points) Use the integral test to prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ (this is known as the $p$-series test). Caution: Do the case for $p = 1$ separately.

3. (6 points)

(a) Find the first three non-zero terms in the Taylor series for $f(z) = \frac{z^2}{z^2 + 2}$ about a point $z_0 = 3$.

(b) Find the region of convergence for the Taylor series.

4. (6 points) Find the Maclaurin series for the following functions. You should notice that, by putting the first two series together, you will get the third series, proving that Euler’s formula is true!

(a) $f(z) = \cos(z)$

(b) $f(z) = i\sin(z)$

(c) $f(z) = e^{iz}$

5. (6 points) Imagine that an electron of mass $m$ is trapped in a one-dimensional potential of the form

$U(x) = A \left[1 - \cos(\alpha x)\right]$,

where $A$ and $\alpha$ are constants. If the electron were in a harmonic potential of the form $U = \frac{1}{2}m\omega^2 x^2$, it would simply oscillate sinusoidally with an angular frequency $\omega$. But this isn’t in a harmonic potential. But if the electron is oscillating with a very small amplitude about the $x = 0$ equilibrium point, $x$ will always be very tiny. If we do a Taylor expansion of some function about $x = 0$, the $x$ term will be small for small values of $x$. But the $x^2$ will be even smaller (compare 0.0001 to 0.0001²). The $x^3$ term will be even smaller than that, and so on. So if we know that $x$ will always be small, higher order terms will matter less than lower order terms.

(a) Write $U(x)$, the anharmonic potential, as a Taylor series expanded about $x = 0$.

(b) If you drop terms higher than $n = 2$, you end up with a harmonic oscillator potential. If the electron oscillates with a very small amplitude, what will the angular frequency of the oscillation be?

6. (12 points) Consider the function

$$f(z) = \frac{\sin(z)}{(z - i)^2}.$$ 

At the location $z = i$, this has a singularity. If we wrote this as a Taylor series about some point $z_0$, the Taylor series would diverge beyond some radius. But a Laurent series expanded around the point
\( z_0 = i \) will converge everywhere except right at the singularity. A Laurent series is similar to a Taylor series, but in addition to having terms of the form \((z - z_0)^n\), it has terms of the form \((z - z_0)^{-n}\). There are line integrals which can be used to calculate the coefficients in the series. But usually you can do a transformation which makes it possible to generate a Laurent series using the simpler methods of a Taylor series. Follow the steps below to do that with the above function.

(a) We want to do our series expansion about the point where the singularity occurs - this will make our Laurent series converge everywhere. So to make the expansion easier, first let \( u \equiv z - i \) and write the function in terms of \( u \) rather than \( z \). (This will move the singularity to the origin.)

(b) Now you should have the sine of something over \( u \) to some power. Well, let’s ignore the \( u \) to some power part, and write a Taylor series of the sine part about the singular point \( u = 0 \). To get the coefficients, write the sines and cosines in terms of complex exponentials (we did that in the last homework). Note that the sine doesn’t have a singularity, so this Taylor series is good everywhere.

(c) Now multiply your Taylor series by the part we left off - the one over \( u \) to some power part. Now you have a series that looks a lot like your Taylor series, but it now has negative powers of \( u \).

(d) Convert back to \( z \). Now you have the Laurent series for this equation - and it converges everywhere except right at the singularity! Yeah!

**For your further enlightenment** (0 points) Use a computer to plot the function in the problem above. Then plot the Laurent series you found (out to, say, about 30 terms). How do they compare? Now use the computer to calculate and plot a Taylor series expansion of the function about \( z = 0 \). Compare the region of convergence for the two series - are they what you expected them to be?