Eigenvectors and Eigenfunctions  OK. I have some complaints about the books discussion of Eigenvalues\(^1\). So I’m going to give a brief intro here - you can read more on Wikipedia.

Eigenvectors and eigenfunctions are defined by the solutions to the following problem:

\[ \textbf{A} \Psi = \lambda \Psi, \]

Here \( \textbf{A} \) is a linear operator, \( \Psi \) is an eigenfunction of the linear operator, and \( \lambda \) is the eigenvalue of the eigenfunctions. (Note the similarities with linear algebra, where we had a similar equation except that \( \textbf{A} \) was a matrix and \( \lambda \) was an eigenvector.)

What is a linear operator, you ask? It is something that, when it acts on \( \Psi \), will make only linear terms. For example, if \( \Psi \) is a function of \( t \), then

\[ \textbf{A} = 4 \frac{\partial^2}{\partial t^2} + 1 \]

is an example of a linear operator. When we “operate” with \( \textbf{A} \) on \( \Psi \), we get

\[ \textbf{A} \Psi = 4 \Psi_{tt} + \Psi. \]

Each term in this is a linear term.

OK, now that we know what a linear operator is, let’s look at the form of the eigenvalue/eigenfunction problem again:

\[ \textbf{A} \Psi = \lambda \Psi. \]

Notice that what this problem says is that when we operate on \( \Psi \) with \( \textbf{A} \), we simply get \( \Psi \) back times a constant. That’s why it’s called an eigenvalue/eigenfunction problem - “eigen” is a German prefix meaning “self.” Any function \( \Psi \) which satisfies this equation is an “eigenfunction” of this equation. And the value of \( \lambda \) that makes that eigenfunction fit the equation is known as the “eigenvalue” of that function.

Let’s say I want to know the eigenfunctions and eigenvalues of the operator \( \textbf{A} = 4(\partial^2/\partial t^2) + 1 \) subject to the boundary conditions \( \Psi(0) = \Psi(1) = 0 \). The first thing I will do is plug the operator into the equation \( \textbf{A} \Psi = \lambda \Psi \) and find the solutions. The equation I get is

\[ 4 \Psi_{tt} + 2 \Psi = \lambda \Psi. \]

Using standard methods for ODEs, we can find that the solution to this equation is

\[ \Psi = C \cos \left( \sqrt{\frac{2 - \lambda}{4}} t \right) + D \sin \left( \sqrt{\frac{2 - \lambda}{4}} t \right). \]

When I apply my boundary conditions, I find that \( C \) must be zero, and \( \sqrt{(2 - \lambda)/4} = n\pi \). So the solutions to this equation, subject to these boundary conditions, have the form

\[ \Psi = D \sin(n\pi t). \]

These are the eigenfunctions for this operator subject to the given boundary conditions. The eigenvalues that go with them are just the \( \lambda \):

\[ \lambda_n = 2 - 4n^2\pi^2 \]

\(^1\)My first complaint is that we are free to choose many forms for our separation constant, and they don’t actually make the connection between the separation constant and the eigenvalue. In the books example, they set both sides of the equation equal to \( -\lambda^2 \), and then said that the values of \( \lambda \) which fit the boundary conditions were the eigenvalues. But they could have set both sides equal to \( \lambda \) or \( -\lambda \), or a number of other things. They never explained what was special about \( -\lambda^2 \). My second complaint is that, in addition to not telling you why they chose \( -\lambda^2 \), it turns out that choosing \( -\lambda^2 \) was the wrong thing - it goes against the standard convention of what is meant by eigenvalues in this context.
What does this have to do with solving PDEs? Eigenvalues and eigenfunctions show up when solving PDEs because they are exactly the kind of problem we get when we use separation of variables. For example, if I were solving the wave equation $c^2 u_{xx} = u_{tt}$ by separation of variables, I would look for solutions of the form $u = a(x)b(t)$. Plugging this in, I get

$$c^2 a'' b = ab'' \Rightarrow c^2 \frac{a''}{a} = \frac{b''}{b}.$$ 

Then I determine that each side must be equal to a constant. Let’s call this constant, say, $\lambda$. Then I get two ODEs:

$$a'' = \frac{\lambda}{c^2} a \quad \text{and} \quad b'' = \lambda b.$$ 

Both of those ODEs are in the form of an eigenvalue/eigenfunction problem. The solutions to the $x$ equation are just the eigenfunctions of the operator $A = \frac{\partial^2}{\partial x^2}$, and the solutions to the $t$ equation are the eigenfunctions of the operator $B = \frac{\partial^2}{\partial t^2}$.

You might have noticed that we’ve already been solving these kinds of problems, even before we knew how to talk about them in terms of eigenfunctions and eigenvalues. And being able to solve problems is what matters most to me. But I’d like you to know the accepted definitions and terminology in case you ever need to impress a math major or help someone answer a question on eigenvalues on a game show. And, more to the point, using this terminology helps you make connections with linear algebra, which should help you both understand and remember the techniques we use that involve eigenfunctions.

1. (8 points) Find the eigenfunctions and the corresponding eigenvalues for the linear operator

$$A = \frac{\partial^2}{\partial t^2}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$ 

2. (10 points) Consider again the equation we derived for heat flowing through a rod

$$T_t = \frac{k}{\rho c} T_{xx}.$$ 

This equation tells us that the rate at which the temperature at location $x$ changes depends on the curvature of the temperature - if the slopes are different on either side of that piece of the rod, the heat flowing in one side won’t equal the heat flowing out the other, the thermal energy in that piece of the rod will change with time, and the temperature will change. Now let’s consider what would happen if heat was allowed to flow out the side of the rod. To model this mathematically, we’ll add a term that represents the temperature changing because heat is flowing out of each slice of the rod as well as flowing from one slice to another. The simplest term that describes that kind of behavior is one that is just proportional to the local temperature of the rod, giving us the equation

$$T_t = \frac{k}{\rho c} T_{xx} - \beta T.$$ 

Note that this says that if the a slice of the rod is hotter, heat flows out the side faster, causing the temperature to decrease at a greater rate.

Using nothing more than separation of variables, find the general solution to this PDE subject to the boundary conditions $T(0,t) = T(L,t) = 0$.

3. (16 points) Now let’s take another approach to solving this equation . . .
(a) Consider the limit in which $\beta$ is so large that we can ignore the $T_{xx}$ term in the equation (i.e. heat flows out the side of the rod much faster than it flows through the rod). In this case, each piece of the rod acts independently, and it turns out that the solution to the problem will be just be some unknown function of $x$ times a function of $t$. In other words, you will find that $T(x, t) = f(x)g(t)$. Find $g(t)$.

Hint: Because this problem only involves derivatives with respect to one independent variable, it is really more like an ODE than a PDE problem. When you do separation of variables, the $f(x)$ part will cancel out entirely, meaning that it can be anything. Then you'll just be left with one ODE to solve for $g(t)$. Your solution to the full equation will just be that arbitrary $f(x)$ times the $g(t)$ you found. And, like an ODE problem, there will be no summation in your solution. You might think of it as an ODE problem where $f(x)$ is the arbitrary constant in front of your solution - OK, it's not a constant, but it's constant with respect to $t$!

(b) Show that $f(x)$ is just the initial condition $T(x, 0)$.

(c) Now let’s consider the case where the $-\beta T$ term and the $T_{xx}$ term both matter. To do this, we'll try to extract the physics of heat flowing out of the side by defining some new function $w(x, t)$ such that $T(x, t)$ is equal to $w(x, t)g(t)$ where $g(t)$ is what you found in part (a). Find the differential equation that $w(x, t)$ solves.

(d) Solve the differential equation to find $w(x, t)$, and then transform back to find the general solution for the PDE.

(e) Apply the boundary conditions. You should have the same thing as the answer you found in the last problem.

(f) Now assume that our bar was initially at a temperature $T_b$ everywhere. Find $T(x, t)$.

4. Consider the PDE below, known as the “diffusion-convection” equation. The extra term in this equation describes a moving medium - if heat is diffusing through air, but the air particles themselves are moving at a velocity $v$, then the temperature at a given point changes because of heat diffusion, but also because the air at that point is moving out and new air, at a different temperature, is moving in.

$$u_t = \gamma u_{xx} - vu_x$$

Here $\gamma$ and $v$ are constants.

(a) First solve the equation we would get if there were no diffusion term: $g_t = -vg_x$. Do this by separation of variables. You should get something that looks like

$$Ae^{\beta(x-B)}$$

where $A$ and $\beta$ are arbitrary constants and $B$ is a constant which is NOT arbitrary but I’m not going to give to you - you’ll have to find out what it is by solving the problem.

(b) Now we’ll use the solution from (a) to try to simplify our PDE. Let $u = e^{\beta(x-B)}w(x, t)$ and find the PDE that $w(x, t)$ satisfies.

(c) When you did part (b), you shouldn’t have gotten the diffusion equation. We tried a trick, and it didn’t work. Or did it? Remember that $\beta$ is an arbitrary constant. We can’t turn this PDE into the diffusion equation just by making a good choice for $\beta$. But we can get an equation that looks kind of like the one we solved in problems 2 and 3 above (but with a different constant and a different sign in front of the “extra” term). What does $\beta$ need to be for that to happen?

(d) Using what we learned on this problem and problem 3, we can write $u(x, t) = a(x, t)b(x, t)$ such that $b(x, t)$ solves the standard 1D diffusion equation. What is $a(x, t)$?

Notice that on the last problem, you did the same thing we did before, but it didn’t fix the problem in one step. The method we used is not a “plug and chug” method, it’s a guess and try (and try to be clever) method.