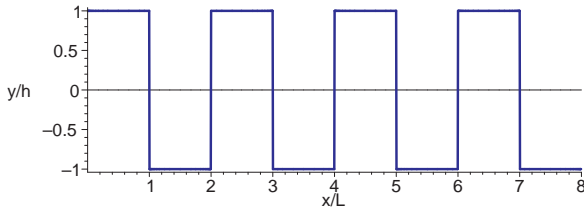


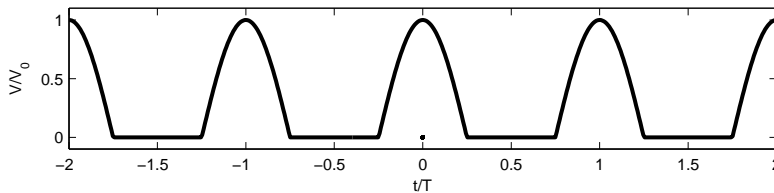
1. (8 pts) A square wave is traveling in the  $+x$  direction down an infinitely long string. The string has a linear mass density  $\mu$  and is under a tension  $T$ . At time  $t = 0$  the wave is described by the figure and equation below.



$$y = \begin{cases} h & \text{for } 0 < x < L, 2L < x < 3L, \text{ etc.} \\ -h & \text{for } L < x < 2L, 3L < x < 4L, \text{ etc.} \end{cases}$$

Let's write this function as a sum of sines and cosines. Let's do a Fourier Transform!

- (a) What are  $\lambda_0$  and  $k_0$ ?
  - (b) Show that  $a_n = 0$  for all even values of  $n > 0$ , and find  $a_n$  for any arbitrary odd  $n > 0$ .
  - (c) What are  $b_0$  and  $b_n$  for  $n > 0$ ? (You can answer this one with symmetry arguments.)
2. (4 pts) We have an equation for the time-averaged power carried by a sine wave traveling down a string. For a more complicated wave, the power is just the sum of the powers carried by each of the sine waves that make it up. Show that the square wave above carries infinite time-averaged power. (This is one reason that you can't ever make a true square wave on a string — you can only approximate one.)
3. (4 pts) The power that comes out of the electrical outlets in your home is AC or "alternating current" power, characterized by a voltage that oscillates sinusoidally. Unfortunately, most electronic devices require a constant voltage (known as DC or "direct current" power). One way to generate DC power from AC power is to use a device called a diode to "kill" the negative part of the sine wave, resulting in a wave like the one shown below. Then the wave is filtered to keep only the DC component (i.e. the  $b_0$  term of your Fourier series). The wave below is known as a "half-wave rectified" wave ("rectified" because it is only positive, and "half" because only half of the wave is left). If I wanted to make a 5 V DC power supply this way, what should the amplitude of the oscillating sine wave be before it is rectified?



4. (10 pts) Some guitar amplifiers produce a heavy distortion of the signal from the guitar by applying this same type of half-wave rectification discussed in the previous problem. This type of distortion is very popular in heavy metal music. For simplicity let's imagine that a guitar is generating a sine wave of the form  $V_{guit} = V_0 \cos(\omega t)$ . This signal is then rectified to make a wave like the figure in the previous problem. Give all of your answers in terms of  $V_0$  and  $\omega$ .

- (a) What is  $b_0$  for the rectified signal?
- (b) Give a good argument as to why all of the  $a_n$  terms should be zero.
- (c) Find  $b_1$ . Hint, instead of integrating from 0 to  $T$ , integrate from  $-T/2$  to  $T/2$ , then use the following integral:

$$\int_{-\pi/2}^{\pi/2} \cos^2(u) du = \frac{\pi}{2}.$$

- (d) Plug the wave into the appropriate Fourier integral and show that all of the other odd  $b_n$  terms are zero and find an expression for even  $b_n$  terms for  $n > 1$ . (You will need the following integral:

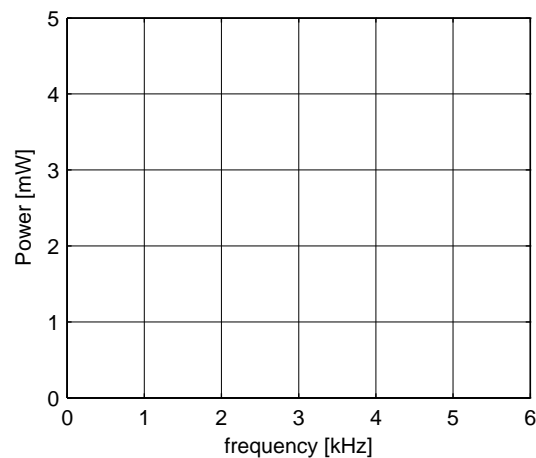
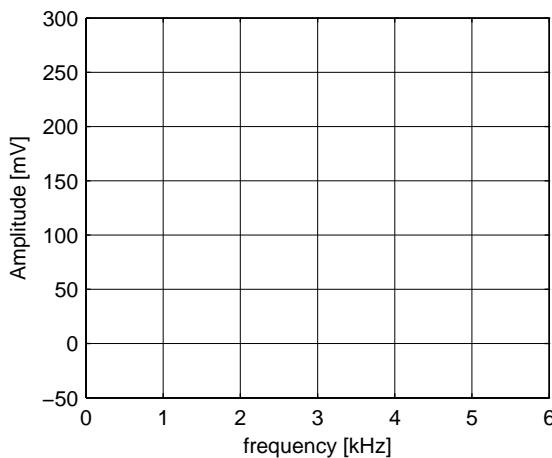
$$\int_{-a}^a \cos(px) \cos(qx) dx = \frac{\sin((p-q)a)}{p-q} + \frac{\sin((p+q)a)}{p+q}.$$

Also note that  $\sin(m\pi/2)$  is equal to zero if  $m$  is even, and is equal to  $(-1)^{(m-1)/2}$  if  $m$  is odd.) To make the grader's life easier, please write your answer as something times  $(-1)^{n/2}/(n^2 - 1)$ .

5. (4 pts) Now lets imagine that we send this rectified signal to a speaker. The power delivered to a speaker by a constant voltage  $V$  (the  $b_0$  term) is  $P = V^2/R$ , and the time-averaged power delivered by a sine wave is equal to

$$P = \frac{A^2}{2R}$$

where  $R$  is the impedance of the speaker (which has units of Ohms) and  $A$  is the amplitude of the sine wave (which has units of Volts - one Volt squared per Ohm is a Watt). Plot the magnitude of the amplitude ( $\sqrt{a_n^2 + b_n^2}$ ) and power spectrum of the rectified wave on the graphs below. Assume that  $V_0 = 0.5$  V, that the frequency of the wave generated by the guitar is 1 kHz, and that  $R = 8$  Ohms (a typical value for a speaker in a guitar amplifier).



**Extra problems I recommend you work (not to be turned in)**

- We usually call the standard electrical lines in our home “120 V” lines, even though the amplitude of the oscillating voltage is 170 V. This is because the electrical power delivered to a resistive load depends not on the voltage across the load, but the voltage squared. Show that a wave with an amplitude of 170 V has an *rms* amplitude of 120 V.
- It turns out that the total time-averaged power delivered to the speaker by any arbitrary wave is equal to  $P = V_{rms}^2/R$ . Show that if you calculate the rms voltage of the rectified wave, it is just equal to the sum of the rms voltages of all of the sine waves.
- We discussed the power reflected by a wave when the string properties change. Imagine that the square wave above is traveling on a string with a linear mass density  $\mu_1$  and a tension  $T$ . At some point the string's mass density changes to  $\mu_2$ . What will the reflected wave look like?
- Use a Fourier transform to show that  $\sin(\pi x/L) \sin(10\pi x/L)$  is just the sum of two sine waves, and find the frequency and amplitude of those sine waves.
- A simple and often used electrical circuit is a low-pass filter consisting of a resistor followed by a capacitor connected to ground. This type of filter is often characterized by its “time constant”  $\tau$ . If a sine wave with angular frequency  $\omega$  passes through this filter, its amplitude drops by a factor of  $\sqrt{1/(\omega^2\tau^2 + 1)}$ , and the phase of the sine wave gets shifted by  $\arctan(-\omega\tau/2)$ . Imagine that I could make a similar filter for waves on a string. After our square wave passed through a filter with a time constant  $\tau$ , how much power would be transmitted by the wave?
- Use a computer to plot the wave that you would get if you put an electrical square wave through a low-pass filter consisting of a resistor and a capacitor to ground.