

Example: Adding three sine waves together using complex exponentials

Imagine that three boats turn on their engines, creating sine waves on the surface of the water. If boat #1 were making waves alone, the displacement of the water where I am swimming would be given by  $y_1(t) = A_1 \sin(\omega t + \phi_1)$ , where  $A_1 = 2.5$  mm and  $\phi_1 = 0.7$  rad. If boat #2 were making waves alone, the displacement of the water where I am swimming would be given by  $y_2(t) = A_2 \sin(\omega t + \phi_2)$ , where  $A_2 = 3.3$  mm and  $\phi_2 = 1.7$  rad. If boat #3 were making waves alone, the displacement of the water where I am swimming would be given by  $y_3(t) = A_3 \sin(\omega t + \phi_3)$ , where  $A_3 = 1.9$  mm and  $\phi_3 = -1.1$  rad. Together they make a sine wave  $y_{total}(t) = A_{total} \sin(\omega t + \phi_{total})$ . What are  $A_{total}$  and  $\phi_{total}$ ?

To do this, lets first write down the three functions we are adding together:

$$\begin{aligned}y_1(t) &= A_1 \sin(\omega t + \phi_1) \\y_2(t) &= A_2 \sin(\omega t + \phi_2) \\y_3(t) &= A_3 \sin(\omega t + \phi_3)\end{aligned}$$

We get  $y_{total}$  simply by adding these three functions together. But finding the amplitude and phase of the resulting sine wave is difficult unless we use complex exponentials. So lets do that. First, we write down three complex functions which are not equal to, but represent the functions above:

$$\begin{aligned}\tilde{y}_1(t) &= A_1 e^{i(\omega t + \phi_1)} \\ \tilde{y}_2(t) &= A_2 e^{i(\omega t + \phi_2)} \\ \tilde{y}_3(t) &= A_3 e^{i(\omega t + \phi_3)}\end{aligned}$$

These functions are not the same as the ones above, but Euler's formula tells us that the imaginary part of each of these functions is equal to each of the real functions above that represent the real waves on the water. So if I add these functions together, the imaginary part of the sum will be equal to the real sum of the real functions. In other words:m

$$y_{total} = y_1 + y_2 + y_3 = \Im(\tilde{y}_1) + \Im(\tilde{y}_2) + \Im(\tilde{y}_3) = \Im(\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3).$$

So far, this sum doesn't look any simpler to work with than the sum of sine waves we started with. But the great part about exponentials is that we can separate the  $\omega t$  and the  $\phi$  parts:

$$Ae^{i(\omega t + \phi)} = Ae^{i\phi} e^{i\omega t}.$$

We can rewrite this as

$$\tilde{A}e^{i\omega t}$$

using the definition that  $\tilde{A} \equiv Ae^{i\phi}$ . Applying this to my complex functions, I get

$$\begin{aligned}\tilde{y}_1(t) &= \tilde{A}_1 e^{i\omega t} \\ \tilde{y}_2(t) &= \tilde{A}_2 e^{i\omega t} \\ \tilde{y}_3(t) &= \tilde{A}_3 e^{i\omega t}\end{aligned}$$

where

$$\begin{aligned}\tilde{A}_1 &= A_1[\cos(\phi_1) + i \sin(\phi_1)] = 1.91211 \text{ mm} + i \cdot 1.61054 \text{ mm} \\ \tilde{A}_2 &= A_2[\cos(\phi_2) + i \sin(\phi_2)] = -0.42519 \text{ mm} + i \cdot 3.27249 \text{ mm} \\ \tilde{A}_3 &= A_3[\cos(\phi_3) + i \sin(\phi_3)] = 0.86183 \text{ mm} - i \cdot 1.69329 \text{ mm}\end{aligned}$$

and my sum is now just

$$y_{total} = \Im(\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3) = \Im([\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3]e^{i\omega t}) = \Im(\tilde{A}_{total}e^{i\omega t}) = \Im([2.34875 + i \cdot 3.18974]e^{i\omega t}).$$

So  $\tilde{A}_{total} = 2.34875 + i \cdot 3.18974$ . But remember,  $\tilde{A}_{total} = A_{total}e^{i\phi_{total}}$ . So

$$A_{total} = |\tilde{A}_{total}| = \sqrt{[\Re(\tilde{A}_{total})]^2 + [\Im(\tilde{A}_{total})]^2} = \sqrt{\tilde{A}_{total}^* \tilde{A}_{total}} = \boxed{3.96 \text{ mm}}.$$

The phase is just the angle that  $\tilde{A}_{total}$  makes relative to the real axis in the complex plane. So

$$\phi_{total} = \arctan\left(\frac{\Im(\tilde{A}_{total})}{\Re(\tilde{A}_{total})}\right) = 0.936 \text{ rad}.$$

Now, before I make that my final answer, I have to remember that the arctan of  $\theta$  is equal to the arctan of  $\theta + \pi$ . So I think for a minute and realize that since both the real and imaginary parts of  $\tilde{A}_{total}$  are positive, the phase angle must be between 0 and  $\pi/2$  radians. This answer fits the bill, so it is correct. Otherwise, I'd have to add or subtract  $\pi$ . Therefore

$$\phi_{total} = \boxed{0.936 \text{ rad}}.$$

I plotted this on the computer, and it looks like I did everything right, because it works!