

1. (6 pts) In the quantum model of conductivity, the atoms accelerate in the electric field for an average time τ before colliding with an ion and losing kinetic energy.

- (a) The “random” component of a typical electron’s velocity (which is equal to the Fermi velocity v_F) is much much greater than the drift velocity. As such, this is the velocity which determines how often an electron collides with an ion. Assuming that when an electron hits something it loses all of its kinetic energy, show that the maximum drift velocity that an electron obtains is, on average, equal to

$$v_d = \frac{eE\lambda}{mv_F},$$

where e is the charge of an electron, m is the mass of an electron, and λ is the mean free path that an electron travels before colliding with something. In fact the electron doesn’t always come to a complete stop at each collision. Because of this, without further proof I will state that the value that you found above is precisely equal to the average drift velocity of the electrons v_d .

- (b) If I place a voltage V across a rectangular piece of copper which is a length L long, what is the electric field E inside the copper?
- (c) Remembering Ohm’s law: $V = IR$, and remembering that for a rectangular conductor of length L with a cross sectional area A the resistance is $R = \rho L/A$, use the drift velocity you found to show that the resistivity of a metal should be equal to

$$\rho = \frac{mv_F}{ne^2\lambda},$$

where n is the number of electrons per unit volume in the metal.

Now let’s use this equation to calculate the resistivity of copper at room temperature (300 K). The only empirical data that we will use is the mass of a copper atom (63.546 amu), the density of solid copper ($8.96 \times 10^3 \text{ kg/m}^3$), the Debye temperature of copper (309 K, as deduced from measurements of the heat capacity of copper), and the assumption that each copper atom contributes one conduction electron. (The Debye temperature is found by fitting the measured heat capacity as a function on temperature to the Debye theory of the heat capacity of solids, and is roughly speaking the temperature at which the heat capacity of a solid begins to drop below $3R$.)

2. (4 pts) The first thing that we need to find is n , the number density of conduction electrons (the number of conduction electrons per unit volume). Use the density and atomic weight of copper to calculate the number of lattice ions per unit volume. We’ll call this n_{ion} . If we assume that one conduction electron is donated by each copper atom, then this is also the density of conduction electrons.
3. (4 pts) The next thing that we need is the Fermi velocity v_F for copper. Well, what are you waiting for? Find it!

The final piece to find is λ . This will take a little more work. To do this we first need to find the effective “cross section” of the scatterers. The cross section is the “effective” area of the scatterer. In other words, if we think of the ions as little targets, the cross section is the area of the target. If the area is big, the electrons will collide with them more often. Remember, the electrons diffract around the ions and wouldn’t scatter if it weren’t for the fact that thermal fluctuations move the ions from their equilibrium positions and destroy the perfect periodicity of the lattice.

4. (7 pts) A solid which is at temperatures well above the Einstein temperature (the temperature at which the heat capacity starts to drop) has 6 degrees of freedom — 3 translational and 3 potential energy degrees of freedom. Each degree of freedom contains an average energy equal to $k_B T/2$, where k_B is Boltzmann’s constant. This

is why the molar heat capacity of a solid is just $3R$. This means that the average potential energy due to displacement of an ion from its equilibrium position in the x direction is just

$$\frac{1}{2}K\bar{x}^2 = \frac{1}{2}k_B T$$

where K is the “spring constant” which tells us the force on the ion as a function of displacement, and \bar{x}^2 is the average value for x^2 . The equations for the y and z directions are the same. If we define the z axis to be the direction in which a particular conduction electron is moving, then the size of the “target” that the electron can hit just depends on the x and y components of displacement. In other words, each ion acts like a little scattering target with an area equal to $\sigma = \pi r^2 = \pi(\bar{x}^2 + \bar{y}^2)$. This area is known as the scattering cross section. If we then assume that the ion moves just as much in the y direction as the x direction, then $\bar{y}^2 = \bar{x}^2$, and $\bar{r}^2 = 2\bar{x}^2$. To find the conductivity we need to know this cross section, and to find the cross section we need to know K .

(a) We can get K from the Einstein temperature T_E , which is just, as it turns out, always equal to 0.774 times the Debye temperature. In Einstein’s model of the specific heat of solids the atoms move in a harmonic potential, and the Einstein temperature is the temperature at which each degree of freedom has about one quanta of excitation: $k_B T_E = \hbar\omega$ where ω is the natural oscillation frequency for the ion’s motion. For a classical harmonic oscillator, ω is just equal to $\sqrt{K/M}$ where M is the mass of the thing that is oscillating (an ion in this case). So from T_E we should be able to get ω , and from that we should be able to get K . What is K for copper?

(b) Using your answer from part (a), find the scattering cross section $\sigma = \pi r^2$ at room temperature (300 K).

5. (5 pts) Using the scattering cross section you have found, we can now find the mean free path, λ . This is the distance that an electron travels, on average, before it scatters. Sometimes an electron won’t travel very far before scattering. Other times it will “slip through the cracks” and travel a long distance. On average, however, the electron travels a distance such that if we add together the cross sections of all of the ions it has past, it will add up to the total cross sectional area of the piece of metal the electron is traveling through. The number of ions that the electron passes as it travels a distance λ is just equal to $N_{passed} = n_{ion}A\lambda$ where A is the cross sectional area of the metal. But if λ is the mean free path, then the sum of the cross section areas of all of the ions in that volume must equal the total cross section of the tube of ions we are considering: $N_{passed}\sigma = A$.

(a) Find an equation for the mean free path λ as a function of σ and n_{ion} .

(b) What is the mean free path for conduction electrons in a piece of copper.

6. (4 pts) Now put it all together and find the resistivity of copper. The measured value is $1.72 \times 10^{-8} \Omega \cdot \text{m}$. You should get something within 20% of that. How’d you do?