You might find the following integral useful on this assignment. Here \( n \) is an integer.

\[
\int_0^{n\pi} x \sin^2(x) \, dx = \frac{n^2\pi^2}{4}
\]

1. (3 pts) Let’s look at some operators acting on the function \( f(x) = x^4 \). Calculate \( \dot{x} \hat{p} f(x) \) and \( \dot{\hat{p}} x f(x) \) and show that they are not the same (i.e., \( \dot{x} \) and \( \dot{\hat{p}} \) do not commute).

2. (3 pts) (a) Use equation 5.46 and the information in section 5.7 to find the full time-dependent wave functions of the \( n^{th} \) energy eigenstate of a particle in a one-dimensional box \( \Psi_n \), as a function of \( n, x, t, L \) (the size of the box), \( E \) (the energy of the eigenstate), and fundamental constants. (b) Show that \( P(x, t) \, dx \) for these energy eigenstates does not change in time.

3. (5 pts) Use the full time-dependent waveforms to find the expectation values \( \langle x \rangle, \langle p \rangle, \langle p^2 \rangle, \langle E \rangle \), and \( \langle KE \rangle \) for the \( n^{th} \) state of an infinite square well as a function of time. Remember that

\[
\hat{E} = \frac{\hbar}{i} \frac{\partial}{\partial t} \quad \text{and} \quad KE = \frac{p^2}{2m}.
\]

Also remember that when an operator is squared, that means that you operate twice. (Note that time should cancel out in all of these expectation values. Energy eigenstates — also known as stationary states — have this unique property that although the wave function evolves in time, it does it in such a simple way that no physically measurable property changes in time.)

4. (4 pts) A pure momentum state for a free particle has a wave function of the form:

\[
\Psi = A e^{i(kx - \omega t)}.
\]

These states are not normalizable, so they can’t describe a real particle. But they are still very useful. Often we will describe a real particle as a sum of these states (using a complex Fourier transform). So it’s worth thinking about them a bit. (a) Use the criterion in equation 5.34 to prove that a wave function of this form is an eigenstate of \( \hat{E}, \hat{p} \), and \( KE \). (b) Show that it is not an eigenstate of or \( \dot{x} \).

5. (5 pts) Imagine an electron traveling in free space with a wave function given by \( \Psi = A \left[ e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)} \right] \) where \( A, k_1, k_2, \omega_1, \) and \( \omega_2 \) are constants. (a) What is the probability function \( P(x, t) \)? Write it in such a way that it is obvious that it is real (not complex). (b) What are the expectation values \( \langle k \rangle, \langle \omega \rangle, \langle p \rangle \) and \( \langle E \rangle \)? (Hint — you don’t have to do an integral, but you can if you want to in order to check your intuition).

For the next few we will consider a particle in an infinite square well of length \( L \) which has been prepared in such a way that its wave function is a superposition of two energy eigenstates. At time \( t = 0 \) the wave function inside the well is equal to

\[
\Psi(x, t = 0) = A \left[ \sin \left( \frac{\pi x}{L} \right) + i \sin \left( \frac{2\pi x}{L} \right) \right]
\]

where \( A \) is a real constant.

6. (4 pts) (a) What is \( A \)? (b) If I hadn’t told you that \( A \) was real (i.e., if it could be complex), what other values could it have? (c) If we let \( A \) be complex, how will that change \( P(x) \)?

7. (3 pts) What is the wave function equal to at some arbitrary time \( t \)?

8. (3 pts) What is \( \langle E \rangle \) at some moment in time \( t \)? If you think about this for a moment, you might find an argument which lets you find \( \langle E \rangle \) without doing an integral.
Extra problems I recommend you work (not to be turned in)

- If we measured the location of the particle in problems 5 through 7 at time $t = 0$, what would be the probability of finding it between $x = 0$ and $x = L/2$?

- How much could your answer to the above have changed if we had allowed $A$ to be complex?

- If we measure the location of the particle at time $t = mL^2/2\pi\hbar$, what would be the probability of finding it between $x = 0$ and $x = L/2$?

- At what moments in time is $P(x)dx$ exactly the same as it is at time $t = 0$. This is known as a recurrence.