1. (6 points) (a) Write the symbol that represents the full derivative of \( f \) with respect to \( x \). (b) Write the symbol that represents the partial derivative of \( f \) with respect to \( x \). (c) What is the difference between a full derivative and a partial derivative?

2. (6 points) Consider the following equations evaluated at the location \( x = y = 0 \). Classify the equations as elliptic, hyperbolic or parabolic and state whether the equation describes diffusion, oscillation, or steady-state phenomena at that location.

(a) \( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial P}{\partial y} = 0 \)

(b) \( \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 1 \)

(c) \( x^2 \frac{\partial^2 T}{\partial x^2} + xy \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial y} = 0 \)

3. (11 points) The diffusion equation is the equation that describes how heat moves through solids, how rarefied gases diffuse, etc. Although the equation is very general, let’s use a specific problem to derive it. For simplicity, we’ll only derive it in one dimension.

Assume that I have a bar of metal of length \( L \) with a cross section \( A \) made of a material with a conductivity \( k \), specific heat \( c \), and mass density \( \rho \). Assume the bar is surrounded by a perfect insulator, so we only have to consider heat flow in the \( x \) direction. In your mind, divide the bar up into little tiny slices of length \( dx \). We’ll label each slice with the distance of it’s left-most side from the left end of the bar. So, for example, the left-most slice is the \( x = 0 \) slice, the next one is the \( x = dx \) slice, and so on.

(a) Now consider how much heat is flowing through an arbitrary slice who’s left edge is a distance \( x \) from the left edge of the bar. In Physics 123 we learned that the rate of heat flow through a bar in steady state is

\[
P = \frac{kA(T_{left} - T_{right})}{L}
\]

where \( T_{left} \) and \( T_{right} \) are the temperatures of the left and right sides of the bar (writing the equation this way assumes that positive \( P \) means heat flow from the left to the right). We can’t apply this equation to our bar because it isn’t necessarily in steady state. But if we consider a tiny slice of the bar, in the limit as \( dx \) goes to zero, the slice will have no mass. So it only takes an infinitesimal amount of heat to change it’s temperature by a finite amount, and we can assume that in this limit our slice will always be infinitesimally close to steady state. So apply this equation not to the bar as a whole, but to the infinitesimal slice (Hint - the length of the slice is \( dx \), the temperature at the left side of the slice is \( T(x) \), and the temperature at the right side is \( T(x + dx) \)). The equation you just found looks like a derivative. So take the limit as \( dx \to 0 \) and find an expression for \( P(x) \) in terms of \( \frac{\partial T}{\partial x} \). Note that this is a partial derivative, because we are assuming that we are not changing \( x \) while we watch the heat flow.

(b) Note that the power you found is a function of \( x \) - it is possible that the power flowing into one side of a slice is different from the power flowing out the other side. The net rate of heat entering a slice will just be the power entering on the left minus the power exiting out the right. But remember from Physics 123 that the net heat entering an object during a certain time is related to the change in temperature of an object during that time \( \Delta T \). The relation is given by the
equation: \( Q = mc\Delta T \). Rewrite this equation but writing \( m \) in terms of the mass density of the metal \( \rho \) as well as \( A \) and \( dx \).

(c) Since the net rate of heat flowing into the slice it is just the power entering the left minus the power entering the right, we can write an equation for it:

\[
\frac{\partial Q}{\partial t} = P(x) - P(x + dx).
\]

Plug the expressions you found for \( Q \) and \( P \) into this equation. Now notice that there is something in the equation that looks like a derivative. Take the limit as \( dx \) goes to zero and turn it into a derivative. Note that this will be a partial derivative because when we take the difference of power in and power out, we measure both powers at the same time. What you should have now should look like the diffusion equation, \( Du_{xx} = u_t \), where \( D \) is a constant and \( u \) is the appropriate dependent variable for the problem.

4. (11 points) Consider the equation you just derived - the diffusion equation.

(a) Write what the equation means in words. Something like, “The rate at which the temperature at location \( x \) changes is proportional to . . . .”

(b) Under what conditions is \( T = A\sin(ax)e^{-bt} \) a solution to the PDE you derived?

(c) Show that if \( T_a(x, t) \) and \( T_b(x, t) \) are solutions to the PDE, \( aT_a + bT_b \) is also a solution (where \( a \) and \( b \) are constants).

(d) Using the steady-state equation back in Physics 123 we were able to show that in a uniform bar in steady state, the temperature changes linearly from \( T_{left} \) to \( T_{right} \) as we moved from one end of the bar to the other. Show that you can get that same equation simply by setting \( \partial T/\partial t \) to zero and solving the diffusion equation subject to the boundary conditions that \( T(0) = T_{left} \) and \( T(L) = T_{right} \).

5. (6 points) Write equations that describe the boundary conditions for a bar of length \( L \), cross sectional area \( A \), and heat conductivity \( k \) under the following conditions:

(a) No heat is allowed to enter or leave at the \( x = 0 \) end. At the \( x = L \) end, we are controlling the temperature so that it stays at \( T_a \).

(b) The side at \( x = 0 \) is immersed in a fluid held at a “nominal” temperature \( T_a \), and the \( x = L \) end is immersed in a fluid held at a “nominal” temperature \( T_b \). Hint, the temperature is pretty close to \( T_a \) and \( T_b \) almost everywhere in the fluids. But because of heat flow in and out of the rod, there are some temperature gradients in parts of the fluids. Assume that the “heat exchange coefficients” are \( h_a \) and \( h_b \).

(c) A resistive heater at the \( x = 0 \) end injects heat into the rod at a rate \( P_a \) and a resistive heater at the \( x = L \) end that injects heat into the wire at a rate \( P_b \).

Extra problem (not graded) Derive the 3D diffusion equation.