1. (15 points) Consider the following initial-value problem. Here $\gamma$, $T_0$, $R$, and $k$ are constants. Note that we would have had troubles using Fourier transforms or sine transforms to solve this one because the initial condition does not go to zero nicely as $x \to \infty$.

\[
\begin{align*}
\text{PDE} & \quad T_{xx} = \gamma T_t & 0 < x < \infty, 0 < t < \infty \\
\text{BC} & \quad T(0, t) = Rt & 0 < t < \infty \\
\text{IC} & \quad T(x, 0) = T_0 \sin kx & 0 < x < \infty
\end{align*}
\]

(a) Take that Laplace transform with respect to time of each side of the equation and the B.C. to convert this PDE problem into an ODE problem.

(b) Find the solution to this ODE, apply the transformed B.C., use the fact that $T(x,0)$ is finite everywhere, and find the solution to the ODE problem with no unknown constants.

(c) Now take the inverse transform of your solution to find $T(x,t)$. Note that for part of this you will need to use the Laplace transformation convolution theorem, and note that if $a$ and $b$ are positive real constants then

\[
\int_0^b \text{erfc} \left( \frac{a}{\sqrt{x}} \right) \, dx = -2a \sqrt{b \pi} e^{-a^2/b} + (2a^2 + b) \text{erfc} \left( \frac{a}{\sqrt{b}} \right).
\]

(d) Let $\gamma = T_0 = k = R = 1$, and plot $T(x,t)$ from $x = 0$ to 30 for $t = 0$ and $t = 2$.

2. (10 points) Back in HW #9 we solved the following problem with a sine transform in $x$:

\[
\begin{align*}
\gamma T_t &= T_{xx} & 0 \leq x < \infty \\
T(x, 0) &= T_C \\
T(0, t) &= T_H
\end{align*}
\]

Solve this same problem but this time using a Laplace transform in $t$.

3. (15 points) Consider a semi-infinite insulated rod which is initially at a temperature of zero everywhere. Then at time $t = 0$ I start driving the temperature at one end such that $T(0, t) = T_A \sin(\omega t)$.

(a) Take the Laplace transform of the PDE ($T_{xx} = \gamma T_t$), and the boundary condition to get an ODE problem.

(b) Find the solution to the ODE, assume that the temperature stays finite as $x \to \infty$, and apply your boundary conditions to find the solution to the transformed problem.

(c) Transform back to find $T(x,t)$. To do this, use the convolution theorem. And don’t bother actually evaluating the convolution integral - just write $T(x,t)$ in terms of the convolution integral.