What is convolution? What is it good for?

Imagine that I have a system that responds to some stimulus. For example, I might have a circuit that produces an output voltage that varies with time as a result of an input voltage that varies with time. Or I might have a camera that makes an image of something, but the image isn’t a perfect reproduction. If the system is linear, we can think of the input as a sum of tiny little impulses. For example, imagine that I have a circuit and I drive it with the voltage plotted below:

I can approximate the drive voltage as the sum of a bunch of little square pulses, as shown below. This representation becomes exact in the limit as the thickness of each pulse goes to zero.

If the system is a linear system, we can find what happens when we drive our circuit with this function by finding what happens for each individual square pulse, and then adding the solution to each square pulse together.

So, to solve a problem, we first think of our function as a sum of little pulses. The pulse that occurs at a time \( \tau \) has a height \( f(\tau) \) and a width \( d\tau \). We can write an individual pulse in terms of Heaviside step functions

\[
pulse(t) = H(x - \tau)H(\tau + d\tau - x)f(\tau),
\]

and we can get an approximation of the original function back by adding up all of the pulses. In the limit as \( d\tau \) goes to zero, this sum becomes an integral and we get the function back exactly. In this limit, the product of the two Heaviside step functions becomes \( \delta(t - \tau)d\tau \), where \( \delta(t) \) is the Dirac delta function. We can see that this is true by doing the “sum,” which in this limit is an integral:

\[
\int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau = f(t).
\]

Think for a minute what this integral is telling us - at each point, the function is just equal to a sum where everything goes to zero except a term which just looks like the function evaluated at that point. The integral above is easy to solve - but what it represents is important. Make sure you understand what it tells us before moving on.
When solving this kind of a problem, the next thing to do is find the response due to a Dirac delta function at some arbitrary time $\tau$. Imagine that we discover that if we put in a delta function at time $0$ we end up getting a response $R(t)$. If our system is linear, it shouldn’t make any difference when the pulse arrives - the response should look the same, just shifted in time. So if our pulse arrived at some time $\tau$, instead of getting a response $R(t)$, we would get a response $R(t - \tau)$.

Of course the size of the response to any pulse is proportional to the height of the pulse. So if a delta function induces a response $R(t - \tau)$, then one of the little pulses from our drive, which has the form $f(\tau)\delta(t - \tau)d\tau$ will produce a response $f(\tau)R(t - \tau)d\tau$. So to find out how the system responds to the drive as a whole, we sum up all of the responses due to the different pulses. Of course, since these pulses are infinitely close together, when I say sum, I really mean integral:

$$\text{response} = \int_{-\infty}^{\infty} f(\tau)R(t - \tau)d\tau.$$ 

So, by knowing how our system responds to an impulse, we can find how it responds to any drive function simply by performing this integral. Integrals of this form are known as “convolution” integrals.

We define a special symbol to describe convolution:

$$f(x) \otimes g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$ 

Note that the convolution integral as defined above - and how it’s defined in your book - has a $1/\sqrt{2\pi}$ which in other sources might not be there. The definition we are using makes the convolution theorem, discussed in the homework below, work out nicer. But when we apply it to real, physical things, we have to multiply our results by $\sqrt{2\pi}$ to get rid of this $1/\sqrt{2\pi}$ and get physical results. Also note that convolution commutes:

$$f(x) \otimes g(x) = g(x) \otimes f(x).$$ 

1. (12 points) Rather than working convolution integrals directly, you can also use Fourier transforms to evaluate convolutions. This is especially useful when I’m working with real, discrete, data, because I can use the speedy Fast Fourier Transform algorithm instead of more time consuming numerical integration. We’ll also see later on that we can use it backwards as well - doing convolutions so that we don’t have to do a Fourier integral. Let’s prove the convolution theorem, which relates convolution to Fourier transforms, by following the steps below.

(a) Write down the definition of a convolution

$$f(x) \otimes g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$ 

(b) Now take the Fourier transform (the complex infinite one) of both sides of the equation (you should now have $\mathcal{F}[f(x) \otimes g(x)]$ on the left and a double integral on the right).

(c) For a double integral, we’re free to choose which integral to do first. Let’s plan on doing the $x$ integral first. So for now, think of $\xi$ as a constant. Then make the transformation $v = x - \xi$, and rewrite the double integral in terms of $v$ and $\xi$ (rather than $x$ and $\xi$).

(d) Now your exponential term should have a “plus sign” in it. Use this to split the exponential into two exponentials multiplied together. Then arrange the integrand so that the terms with $\xi$ in them are lumped together, and the terms with $v$ in them are lumped together.

(e) Note that the $\xi$ terms aren’t affected by $v$, and the $v$ terms aren’t affected by $\xi$. So we can bring one set of terms outside of one of the integral signs (because they are constant in that integral). Use this to write your double integral as the product of two single integrals.
(f) Now rewrite $v$ and $\xi$ in each integral as $x$ (v and $\xi$ are just parameters that get integrated away) and distribute the $1/2\pi$ term equally among the two integrals to make the right-hand-side look like the product of two Fourier transforms:

$$\mathcal{F}[f(x) \otimes g(x)] = \mathcal{F}[f(x)]\mathcal{F}[g(x)].$$

2. (16 points) Some of the highest-bandwidth data lines are optical fibers. To send a digital signal down a fiber, I modulate a light source, typically a laser, to send pulses of light down the fiber. The pulses are then detected by a photodiode. There are limitations on how fast you can send data on a fiber - related to things such as how fast you can modulate the light source, dispersion in the fiber, noise, and the response speed of the detector. If these limitations cause pulses to blur together, the signal can be lost. In this problem we’re going to explore the response of a photodiode.

Photodiodes produce a current which is proportional to the power of the light hitting them, according to the equation $I = AP$ where $A$ is a parameter known as the “responsivity” of the diode. The responsivity only depends on the wavelength of the light hitting the photodiode and can usually be treated as a constant. Since most electrical circuits process voltage signals, not current signals, you need a way to convert the current output of the photodiode to a voltage. A very simple way to do that is to take the output of the photodiode and run it through a resistor to ground, as shown in the circuit on the left below. Then the voltage at the top of the resistor will be $V_{out} = IR = APR$.

![Circuit Diagram]

Unfortunately, photodiodes can’t respond to changing light levels infinitely fast. One reason for this is that photodiodes have capacitance. If I model my photodiode as a current source with a capacitor in parallel, as shown in the circuit diagram on the right above, my circuit above becomes an RC circuit. To solve a problem like this one we write down a differential equation for our circuit where the charge on our capacitor is deposited by the current generated by the light, such that the current $AP(t)$ becomes a drive term in the ODE. We can simplify matters by pretending $P(t)$ is a sine wave to make solving the ODE simpler. Then we can build the real $P(t)$ out of sine waves, and find the real response by adding the different sine wave solutions. But, instead, in this problem we’ll pretend that $P(t)$ is a delta function. Then we’ll build the real $P(t)$ by integrating delta functions and find the real response by integrating the delta function responses. In other words, we’re going to convolve the delta function response with the drive.

It’s interesting to note that the convolution theorem connects these two approaches - solving using an impulse response (convolution) and solving using a Fourier series.

Let’s model our circuit as an RC circuit, like the one above. The response of the photodiode to light causes charge to be placed on the capacitor plates. If I had a delta function pulse of light (an infinite spike in power with an integrated area of 1), it would deposit $A$ times one “unit” of charge on my capacitor, such that immediately after the pulse, the output voltage would equal $(A/C)$ (Don’t let it bother you that this doesn’t have units of voltage. The delta function is like a pulse of power, but the delta function is unitless. It has to be because we will get units from the function that we’re going to convolve with our delta function response.) Before and after the light pulse, the current source (the circle with an arrow in it) doesn’t let any current flow - so it acts like an open circuit circuit - as if it wasn’t even there. That’s one of the beauties of using the convolution method - after I deposit my pulse of charge, I can treat my circuit as if it were a single loop!

(a) Before the pulse there is no charge on the capacitor, and the output voltage will equal zero. Right after the pulse, a charge of $A$ will be on the capacitor resulting in an output voltage of $A/C$.

Then the charge will bleed off and $V_{out}$ will decay back to zero. Find $G(t)$, the output voltage of
our circuit ($V_{out}$) when we drive it with $P(t) = \delta(t)$. Note, when you do a circuit loop, solve the problem, and insert your initial condition of $Q(0) = A$, you should get a decaying exponential. But that solution is only good for $t \geq 0$. Make your solution valid at all times using a Heaviside step function $H(t)$ - and don’t worry about what the step function is equal to at $t = 0$. This moment in time is infinitesimally small and not important for this problem.

(b) Let’s imagine we want to send a pulse of light down our fiber whose power is equal to zero at all times except from $t = a$ to $b$ it is equal to $P_0$. Using Heaviside step functions, write an equation for $P(t)$.

(c) The next step is to convolve $P(t)$ with the delta function response. I’ve found that doing this analytically is very challenging - Mathematica choke up on the integrals involved whether you do the convolution directly or whether you use the convolution theorem. So instead, let’s do things numerically. Set $R = P_0 = A = 1$, $C = 0.1$, $a = 4.5$ and $b = 5.5$. Then make a set of data points $P_k$ for $k = 0$ to 200, the first point corresponding to time $t = 0$ with the data points spaced apart in time by a time of 0.05, and plot $P_k$.

If you are using Mathematica, replace the Heaviside step function in your formula with the UnitStep[x] function. This function is well defined at $x = 0$. Also, you might speed up the rest of this assignment if you use the N[x] function on each term to force Mathematica to convert them to floating point numbers. Also, as you do this problem remember that $N$ and $C$ are protected, so you can’t use them as variables. In my solutions I used a lower case $c$ for the capacitance and $NN$ for the number of points.

(d) Now do the same thing for the response function, making a set of points $G_k$.

(e) Now calculate $c_n$ and $d_n$, the coefficients of the complex discrete Fourier transforms of $P_k$ and $G_k$ respectively, and make a plot of the real and imaginary parts of each of them. Note that a DFT assumes that the signal repeats - so instead of convolving a finite pulse with the response of a delta function, we’re really convolving a periodic array of pulses with a periodic array of delta function responses. But that’s OK, because we’ve made our data set long enough for everything interesting to happen in a single period, and we’re only going to look at the first period (this is why I had you make the pulse centered at $t = 4$ instead of $t = 0$).

(f) Now find the convolution by multiplying $c_n$ and $d_n$ term by term, taking the inverse transform, and multiplying by $\sqrt{200/\pi}$. Remember when we proved the convolution theorem there was an extra factor of $\sqrt{2\pi}$ we had to worry about? Well, this is equivalent to that factor when we use the DFT definition I gave you. Plot the resulting photodiode response - it should look like a square pulse, but with an exponential rise and fall on the sides. This is how your photodiode will respond to the square pulse.

(g) Now increase $C$ to 0.5 and plot the response of the photodiode to the square pulse.

3. (12 points) Imagine that I have a system whose response to a delta function impulse is to go up from zero to a constant value $A$, and to stay there for a time $\tau$:

$$ G(t) = AH(t)H(\tau - t). $$

Now imagine that I drive the system with a sine wave

$$ f(t) = B \sin(\omega t). $$

(a) Write down and solve the convolution integral to find the response of the system to the sine wave.

(b) Now get the same solution using the convolution theorem. Not that when you go to do the Fourier transform of $f$, you will get an integral that doesn’t converge. So, instead, remember that

$$ \sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}, $$

and use that to figure out what two delta functions you need to add together and what you need to multiply them by in order to get something that gives you this if you took its inverse Fourier transform.