1. (10 points) Imagine a square pulse is traveling down an electrical cable. At a particular location I measure the voltage as a function of time and I find that it is equal to zero at all times except from time \( t = -\tau \) to \( \tau \) during which time the voltage is equal to \( V_0 \).

(a) This function can be written in the form:

\[
V(t) = \int_0^\infty a(\omega) \cos(\omega t) d\omega + \int_0^\infty b(\omega) \sin(\omega t) d\omega.
\]

Find \( a(\omega) \) and \( b(\omega) \).

(b) Set \( V_0 = 1 \) and \( \tau = 1 \) and plot the spectrum of this wave (i.e. \( C(\omega) = \sqrt{a(\omega)^2 + b(\omega)^2} \) as a function of \( \omega \)) from \( \omega = 0 \) to \( \omega = 20 \).

(c) Find the inverse transform. Then set \( V_0 = \tau = 1 \) and plot it from \( t = -2 \) to \( 2 \) to see if you get the original function back. Note that if \( b \) is a real constant,

\[
\int_{-\infty}^{\infty} \frac{1}{x} \sin(x) \cos(bx) dx = \frac{\pi}{2} [\text{Sign}(1 - b) + \text{Sign}(1 + b)]
\]

where \( \text{Sign}(x) \) returns the sign of \( x \) (i.e., it returns -1 if \( x \) is negative and +1 if \( x \) is positive).

2. (8 points) Now consider the “half-cycle” pulse below:

![Half-cycle pulse graph](image)

This function \( V(t) \) is equal to \( \cos(\pi t/2) \) when \(-1 < t < 1\), and zero everywhere else.

(a) Do a Fourier transform and find \( a(\omega) \) and \( b(\omega) \) for this function. Note that

\[
\int \cos(ax) \cos(bx) dx = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} + C.
\]

(b) Plot the spectrum \( C(\omega) \) of this function from \( \omega = 0 \) to \( 20 \).

3. (8 points) Calculate the following integrals involving the Heaviside step function \( H(x) \) and the Dirac delta function \( \delta(x) \) by hand:

(a) \( \int_{-\infty}^{\infty} H(x) \sin(x) dx \)

(b) \( \int_{-\infty}^{\infty} H(x + 1)H(1 - x) dx \)
(c) \( \int_{-\infty}^{\infty} 2\delta(x)\,dx \)
(d) \( \int_{-\infty}^{8\pi} \delta(x - \pi/2)[\sin(x) + 1]\,dx \)

4. (4 points) If I try to do a Fourier transform of \( \sin(\alpha t) \), I have a problem. It’s easy to see that \( a(\omega) = 0 \) by symmetry. But when I try to evaluate

\[
b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(\alpha t) \sin(\omega t) \,dt,
\]

I realize that if \( \omega = \alpha \), this integral blows up to infinity. And if \( \omega \neq \alpha \) the integral doesn’t converge. But we can use our intuition - there is only one frequency in this function, so \( b(\omega) \) should be zero for any \( \omega \) not equal to \( \alpha \). So it sounds like our transform is like a delta function. Show that if I let

\[
b(\omega) = C\delta(\omega - \alpha),
\]

where \( C \) is a constant, the inverse transform gives me \( \sin(\alpha t) \). And find what \( C \) is equal to.

Before working the next problem, I want to address a notation issue that might come up in your future. In Farlow’s text, the Fourier sine and cosine transform has a constant \( (1/\pi) \) in front of the integral, but there is no constant in front of the integral in the inverse transform. In analogy with Fourier series, we might expect a factor of two difference in the exponential Fourier transform - such that there would be a constant \( (1/2\pi) \) in front of the transform integral and no constant in the inverse transform. But, instead, Farlow tells us that there is a factor of \( \sqrt{1/2\pi} \) in front of the transform, and the same constant in front of the inverse transform!

All that really matters for most applications is that we get the same thing back after doing a transform followed by an inverse transform. So we can take any definition of a transform/inverse transform pair, multiply one by an arbitrary constant, multiply the other by one over the arbitrary constant, and we’ll still have a valid definition for our transform pair.

I personally like having a \( 1/2\pi \) in front of the transform and no constant in the inverse transform. That way my \( c(\omega) \) actually represents the amplitudes of my complex exponentials. But since complex exponentials are already abstract, I’m not going to complain too much, and just stick with the books definitions - and expect you to as well. But you should be advised that there are different definitions used in different places, different classes, different books, and different applications.

5. (10 points) Consider a Gaussian pulse of light of the form \( E(t) = E_0 e^{-(t/\tau)^2} \).

(a) Find the complex Fourier transform of this, \( \mathcal{F}[E] = F(\xi) \). Note that, if I ask Mathematica to integrate

\[
\int_{-\infty}^{\infty} e^{ax+bx^2} \,dx,
\]

it tells me that, as long as the real part of \( b \) is less than zero, I will get

\[
e^{-a^2/4b} \sqrt{\pi}/\sqrt{-b}.
\]

(b) The uncertainty principle tells us that pulses that last shorter lengths of times tend to contain a greater spread of frequencies, such that \( \Delta t \Delta \xi \) should be greater than or equal to \( 1/2 \). Let’s not calculate \( \Delta t \) and \( \Delta \xi \) precisely, but let’s approximate them with the full-width half-maximum of \( E \) and \( F \). Find the full-width half-maximum of \( E(t) \) and the full-width half-maximum of \( F(\xi) \) and show that their product is independent of the \( \tau \).