1. (6 points) (a) Write the symbol that represents the full derivative of \( f \) with respect to \( x \). (b) Write the symbol that represents the partial derivative of \( f \) with respect to \( x \). (c) What is the difference between a full derivative and a partial derivative?

2. (18 points) We’ll get our chance to derive PDEs soon enough. For now, lets just get familiar working with them. Consider the following PDE:

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.
\]

Here \( v \) is a constant. This equation is known as the two-dimensional linear wave equation.

(a) What are the dependent variables in this PDE? What are the independent variables?

(b) What is the order of this PDE?

(c) Is this a linear PDE?

(d) Consider the function \( f = A \sin(k_xx + k_yy - \omega t) \), where \( A, k_x, k_y, \) and \( \omega \) are constants. Under what conditions is this function a solution to the PDE?

(e) If \( f_A(x, y, t) \) and \( f_B(x, y, t) \) are both solutions to the PDE and \( A \) and \( B \) are constants, under what conditions will \( A f_A + B f_B \) be a solution to the PDE?

(f) Use the chain rule\(^1\) to prove that any function of the form \( f(\vec{k} \cdot \vec{r} - \omega t) \) (where \( \vec{k} = k_x \hat{x} + k_y \hat{y} \) is a vector in any direction but with a magnitude equal to \( \omega/v \)) will solve this PDE. (For example, \( f = \sin(\vec{k} \cdot \vec{r} - \omega t) \) or \( f = \ln |A(\vec{k} \cdot \vec{r} - \omega t)^2| \). Hint: Define \( u = \vec{k} \cdot \vec{r} - \omega t = k_x x + k_y y - \omega t \).

3. (16 points) PDEs show up when you have more than one independent variable. It is also possible to have only one independent variable but more than one dependent variable. In that case, instead of a PDE, you get a set of coupled ODEs (if you have multiple dependent and multiple independent variables, you get coupled PDEs). Before we spend most of the semester playing with PDEs, let’s take a moment to consider an example of coupled ODEs. Consider two beads of mass \( m \) sliding without friction along an infinite wire. They are connected to each other by an ideal, massless spring with a spring constant \( k \) and an equilibrium length \( L \).

(a) Let’s use a coordinate system where position increases as we move to the right. Let’s label the position of the bead on the left, relative to the origin, as \( x_1 \) and the position of the bead on the right relative to the same origin as \( x_2 \). Write down an expression for the force on the left most bead as a function of \( m, k, L, x_1, \) and \( x_2 \).

(b) Now write down the force for the rightmost bead.

(c) Now note that \( a_1 = x_{1tt} \) and \( a_2 = x_{2tt} \), and write down a pair of coupled ODEs that describe the motion of the beads.

(d) One way to solve a system of coupled equations is to make a transformation that decouples them, solve the equations, and then transform back. In this case, we might know to expect that the two masses will oscillate relative to each other, but with a constant “center of mass” motion. So let’s try transforming to the new coordinates \( x_{cm} = (x_1 + x_2)/2 \) and \( x_r = x_2 - x_1 \). Solve these two equations to find \( x_1 \) and \( x_2 \) in terms of \( x_{cm} \) and \( x_r \).

(e) Now plug that transformation into the coupled ODEs and manipulate them to get a pair of uncoupled ODEs for \( x_{cm} \) and \( x_r \) (i.e., one ODE that only involves \( x_{cm} \) and not \( x_r \), and one that only involves \( x_r \) and not \( x_{cm} \)).

(f) Solve the two equations, then transform back to show that the general solution is \( x_1(t) = x_0 + v_0 t + A \sin(\omega t + \phi) \) and \( x_2(t) = x_0 + v_0 t - A \sin(\omega t + \phi) \) where \( A, x_0, v_0, \) and \( \phi \) are arbitrary constants. Also, determine what \( \omega \) must be.

\(^1\)Remember that the chain rule says that \( \partial a / \partial b = (\partial a / \partial c)(\partial c / \partial b) \) or, in our shorthand notation, \( a_b = a_c c_b \).