1. (11 points) Let’s convert the time-dependent Schrödinger’s equation,
\[
- \frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r})\Psi = i\hbar \frac{\partial \Psi}{\partial t}
\]
into the time-independent Schrödinger’s equation.
(a) Write \( \Psi(\vec{r}, t) = \psi(\vec{r}) T(t) \). Then use separation of variables to separate the equation into a PDE that depends only on spatial coordinates and an ODE that only depends on time.
(b) Solve the ODE to find \( T(t) \).

2. (11 points) Let’s play with the equation for the hydrogen atom (ignoring electron and nuclear spin).
(a) Write down the time-independent Schrödinger’s equation in spherical coordinates with a potential \( V = -e^2/4\pi\epsilon_0 r \).
(b) Separate the PDE you found above into an angular PDE and a radial ODE.
(c) Note that the solutions to the angular part are the spherical harmonics, \( Y^m_l(\theta, \phi) \). What limitations are there on \( m \)? How does \( l \) relate to the separation constant you used in part (b)?

I’m not going to make you solve the radial part - it is quite time consuming. But just for your information, by writing the radial part as \( r^l e^{-r/2} L(r) \), you can turn the radial ODE into something that looks like the associated Laguerre differential equation, and your solutions end up being
\[
R(r) = r^l e^{-r/2} L_{n-l}^{2l+1}(r)
\]
(plus some another term that blows up at \( r = \infty \)) where \( L_{n-l}^{2l+1}(r) \) are the generalized Laguerre polynomials and \( n \) is a constant. A cool thing to note, however, is that the solution to the angular part did not depend on the potential. As long as your potential is spherically symmetric (i.e., it only depends on \( r \)) the angular part of the solution to Schrödinger’s equation will always be spherical harmonics!

3. (8 points) Let’s use our skills with differential equations to understand how boundary conditions are related to energy quantization. Imagine that an electron is trapped in an infinite one-dimensional square well. The potential is zero from \( 0 \leq x \leq L \), but is infinite from \(-\infty < x < 0 \) and from \( L < x < \infty \).
(a) Find a general solution to the time-independent Schrödinger equation in one dimension in the region where \( V = 0 \).
(b) Outside of this region, it is unphysical for the wavefunction to be non-zero - it would imply an electron with infinite potential energy. So our boundary conditions are \( \psi(0) = \psi(L) = 0 \). Apply these boundary conditions to find \( E_n \) (the allowed energies) and \( \psi_n \) (the wavefunctions for states of well-defined energy). Note that your states will have an arbitrary constant in front of them. You may recall from Physics 222 that by convention we usually normalize the wavefunction such that \( \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \). But I’m not going to make you do that. Just leave it as an arbitrary constant.
(c) Imagine that an electron is initially fairly well localized to a region of size \( w \) at the very center of the well, and is described by the initial state
\[
\Psi(x, 0) = \begin{cases} A & \text{if } \left( \frac{x}{L} - \frac{w}{L} \right) \leq x \leq \left( \frac{x}{L} + \frac{w}{L} \right), \\ 0 & \text{otherwise} \end{cases}
\]
where \( A \) is a constant. Find an expression for what the wavefunction will be at a time \( t > 0 \). You can make things look nicer if you remember the trig identity (which you can derive using complex exponentials) \( \cos(a + b) - \cos(a - b) = -2 \sin(a) \sin(b) \) and remember that \( \sin(m\pi/2) \) is zero for even values of \( m \), and is \( (-1)^{(m-1)/2} \) for odd \( m \).
(d) The probability distribution of the electron (the probability of being found somewhere per unit length) is given by \( \Psi^*(x)\Psi(x) \). Set \( E_n/\hbar = n^2 \), \( A = L = 1 \), and \( w = 0.1 \), and plot the probability distribution of the electron in part (c) at times \( t = 0.0, 0.1, \) and \( 0.2 \). For simplicity, only include the first 20 non-zero terms in your sum.
4. (10 points) Now let’s consider an electron in a cubical quantum dot. The cube has sides of length \( \ell \). For simplicity let’s assume that the potential inside the dot is zero, and outside it is infinite. What are the allowed energies that an electron can have inside the dot?