1. (8 points) Find the Laplace transform of the following functions. Actually do the integral, don’t just look them up in a table. But you may use the table in the back of your book to check your answers if you want. Assume that $A$ and $B$ are constants.

(a) $f(t) = A^2$
(b) $f(t) = A + Bt$
(c) $f(t) = Ae^{Bt}$
(d) $f(t) = \delta(t - 4)$

2. (12 points) Prove the following properties of Laplace transformations. Assume we are transforming with respect to $t$.

(a) $\mathcal{L}[u] = s\mathcal{L}[u] - u(x, 0)$
(b) $\mathcal{L}[u_t] = s^2\mathcal{L}[u] - su(x, 0) - u_t(x, 0)$
(c) $\mathcal{L}[u_x] = \frac{\partial}{\partial x}\mathcal{L}[u]$
(d) $\mathcal{L}[u_{xx}] = \frac{\partial^2}{\partial^2 x}\mathcal{L}[u]$

3. (10 points) Usually, using the convolution theorem for Laplace transforms is harder than just doing the convolution integral. But just to show that we can do it, consider the following problem . . . Imagine that you have a really big bucket. You dump water into it at a rate $R$ (which has units of liters per second).

(a) If you have an empty bucket at time $t = 0$, how much water will you have in the bucket at a time $t > 0$?

(b) OK, you’re probably thinking right now, “Dr. Durfee, what in the world does this problem have to do with convolutions?” Now I will tell you . . . If you put one unit of water into the bucket very rapidly (i.e. in essentially zero time), that water will stick around forever. So the response of your bucket to this impulse is to hold the water for all time greater than $t = 0$. In other words, $g(t) = 0$ for time $t < 0$ and $g(t) = 1$ for time $t \geq 0$. Now, if we pour water at a continuous rate starting at time $t = 0$, that means that $f(t)$ is just $R$ for $t \geq 0$, and zero for times before we start pouring water. Perform the convolution integral to find how much water there is at time $t > 0$.

(c) Now do the same thing, only use the Laplace transform convolution theorem.

4. (10 points) You have spent all day solving a horrible PDE. You started off this mess by taking the Laplace transform of the PDE. Now you finally have a solution to the transformed equation, and you need to transform it back. So, you are staring at the problem below (where $a$ is a constant), and you can’t find the inverse transform of this beast in the appendix of your book, nor do you want to do a contour integral. So instead, find the inverse transform below using the Laplace transform convolution theorem.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - as}\right].$$