1. (25 points) I have a long tube filled with sodium vapor. The tube is so long that we can pretend that it is infinitely long. All of the sodium atoms are optically pumped into the $F = 1$ hyperfine ground state. Then two counter-propagating laser pulses pass through the gas and drive some of the atoms into the $F = 2$ hyperfine ground state through a 2-photon stimulated Raman process. As a result of this, at time $t = 0$, the distribution of $F = 2$ atoms at time $t = 0$ is given by some function

$$\rho(x, 0) = Ae^{-\alpha x^2}$$

where $A$ and $\alpha$ are positive real constants.

(a) The $F = 2$ atoms will diffuse among the atoms in the other state according to the diffusion equation:

$$D\rho_{xx} = \rho_t,$$

where $D$ is the diffusion constant. Reduce this PDE for $\rho$ to an ODE for $U(\omega, t) = F[\rho]$ by taking the Fourier transform in $x$ of both sides.

Notice that although $U$ is a function of $t$ and $\omega$, there are no derivatives with respect to $\omega$. So we can treat $\omega$ almost like a constant for most of the problem when solving PDE’s this way. Also note that we don’t want to do the transform in $t$ because the transform involves integrals from $-\infty$ to $\infty$, and the problem only covers times greater than zero.

(b) Find a general solution to the ODE.

(c) Without ever bothering to work the integral to transform the initial conditions, let’s just define $Q = F[\rho(x, 0)]$. Find the specific solution to the ODE that fits the transformed boundary conditions. Once again, don’t bother to do an integral, just write your solution in terms of $Q$.

What you should have gotten is $Q$ times something that I’ll call $\hat{g}(\omega, t)$.

(d) To get the solution to the PDE, we need to perform an inverse transform on our ODE solution:

$$\rho(x, t) = F^{-1}[U(t)].$$

When you plug in $U(t)$, you get

$$\rho(x, t) = F^{-1}[Q\hat{g}] = F^{-1}[F[\rho(x, 0)]\hat{g}].$$

This kind of looks like the convolution theorem. To take advantage of that theorem (and only have to do one integral rather than three) let’s define some new function $g$ such that

$$F[g(x, t)] = \hat{g}(\omega, t).$$

What is $g(x, t)$? Note that, if $a > 0$, then

$$\int_{-\infty}^{\infty} e^{-ax^2 + ibx} dx = \sqrt{\frac{\pi}{a}} e^{-b^2/4a}.$$

(e) Now use the convolution theorem to find $\rho(x, t)$. Note that if $a < 0$, then

$$\int_{-\infty}^{\infty} e^{ax^2 + bx + c} dx = \sqrt{\frac{\pi}{-a}} e^{-(b^2/4a) + c}.$$

(f) Set $A = D = \alpha = 1$ and plot $\rho(x, t)$ from $x = -10$ to 10 at times $t = 0, 1, 10, \text{ and } 100$.

(g) Show that the total number of atoms in the $F = 2$ hyperfine ground state is conserved as they diffuse with time. Note that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$
2. (6 points) Now we’re going to solve the following problem

\[ Du_{xx} = u_t \quad -\infty < x < \infty, \quad t > 0 \]

\[ u(x, 0) = \delta(x - x_0) \]

where \( \delta(x) \) is the Dirac delta function. Other than the initial conditions (and the fact that we’re using the dependent variable \( u \) instead of \( \rho \)) this is the same problem we just worked. Note that the initial condition didn’t come into play until we finally got the solution by performing a convolution integral. So solve this problem by convolving your initial condition with the \( g(x, t) \) that you found in the last problem.

The fact that \( g(x, t) \) is independent of the initial conditions gives us a lot of power - we can find \( g(x, t) \) once, and then we can solve any problem for the given PDE just by convolving the initial conditions with \( g(x, t) \). Because \( g(x, t) \) is so special, it has a name - we call it the “kernel” of the problem (for this particular problem - the diffusion equation on an infinite one-dimensional medium - we get what is known as the “heat kernel” or “Gauss’ kernel”). Since what we do with the kernel involves convolution, we can define a new function

\[ G(x, s, t) = \frac{1}{\sqrt{2\pi}} g(x - s, t) \]

which we call the “Green’s function.” Then we get our solution simply by multiplying our Green’s function by \( u(s, 0) \) and integrating over \( s \).

Note that our delta function initial condition resulted in a solution which was just the Green’s function with \( s = x_0 \). Because of this property, the Green’s function is often known as the “impulse-response” function. When we find the Green’s function, what we’ve really found is the solution to the problem if the initial condition is just a delta function centered at \( x = s \). To solve problems with arbitrary initial conditions, we just think of our initial condition as a sum of delta functions, and add them all up by integrating our initial condition times the Green’s function.

3. (9 points) Consider an infinite rod which is not insulated, such that heat can exit through the sides. This rod follows the equation

\[ T_t = DT_{xx} - \alpha T \quad -\infty < x < \infty, \quad t > 0. \]

(a) Find the kernel for this problem.
(b) Find the Green’s function for this problem.
(c) Find the solution to this problem assuming that

\[ T(x, 0) = \delta(x). \]