I’m not going to give you any integral tables this time - there’s too many of them. Feel free to use Mathematica, Wolfram Alpha, Maple, books, whatever, to do your integrals.

1. (20 points) Hermite polynomials show up in the solutions to the quantum mechanical simple harmonic oscillator. The first three of them have the form

\[ H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2 \]

(a) Show that \( H_0(x), H_1(x), \) and \( H_2(x) \) are all orthogonal to each other with respect to the weight function

\[ w = e^{-x^2} \]

on the interval \((-\infty, \infty)\).

(b) What are the norms of \( H_0(x), H_1(x), \) and \( H_2(x) \) with respect to the weight function?

(c) If \( g(x) = x^2 + 1 \), what is the inner product \( \langle H_1, g \rangle \) with respect to the weight function?

(d) I want to write \( f(x) = x^2 + 1 \) as a sum of Hermite polynomials:

\[ x^2 + 1 = \sum_{n=0}^{\infty} A_n H_n(x). \]

What will \( A_0, A_1, \) and \( A_2 \) be?

2. (14 points) Let’s make our own set of orthonormal polynomials of degree \( n \). Let’s call them \( V_n(x) \). I want them to be orthogonal on the region \((0,1)\) with respect to the weight function \( w(x) = 1 \).

(a) The first polynomial, \( V_0(x) \) is of degree 0 - in other words it is just a constant: \( V_0(x) = A \). Find what the constant must be if I want my set to be orthonormal.

(b) The next one must have the form \( V_1(x) = Bx + C \). Find what \( B \) and \( C \) need to be in order for \( ||V_1(x)|| = 1 \) and \( \langle V_0, V_1 \rangle = 0 \).

(c) Now I want to write the function \( f(x) = \sin(\pi x) \) as a sum of my polynomials:

\[ \sin(\pi x) = \sum_{n=0}^{\infty} A_n V_n(x). \]

What will \( A_0 \) and \( A_1 \) be?

3. (6 points) If \( f(x) = \sin(\pi x) \) and \( g(x) = e^{4ix} \), what is \( \langle f, g \rangle \) on the interval \((-1,1)\) with respect to the weight function \( w(x) = 1 \)?