1. (12 points) Use separation of variables to solve Laplace’s equation in spherical coordinates:
   (a) Separate the equation into an ODEs in \( r \) and an equation in \( \theta \) and \( \phi \) that is solved by spherical harmonics.
   (b) Solve the ODE in \( r \).
   (c) Now use separation of variables to find an equation for \( \theta \) and an equation for \( \phi \).
   (d) Solve these equations. To give honor to tradition, let your solution to the \( \phi \) part be complex exponentials (not real exponentials or sines and cosines). Note that sines and cosines will solve the equation but won’t give you Laplace’s spherical harmonics in the next step. A solution with real exponentials in \( \phi \) will change the \( \theta \) equation so it won’t look like Legendre’s equation anymore.
   (e) Assume that the solution is finite at all values of \( \theta \) and \( \phi \) and use your results to find a formula for the spherical harmonics in terms of associated Legendre functions to within a multiplicative constant.

2. (6 points) Write the equation \( f(\theta, \phi) = \delta(\theta - \theta_0)\delta(\phi - \phi_0) \) as a sum of spherical harmonics.

3. (12 points) In Mathematica, the function \texttt{SphericalPlot3D[f[theta,phi],[theta,0,pi],[phi,0,2*pi]]} will plot a surface that, at any given angle, is a distance \( f(\theta, \phi) \) from the origin. It is useful for visualizing spherical harmonics. Use the formula below to generate the following plots using the \texttt{SphericalPlot3D} function:

   \[
   Y_l^m(\theta, \phi) = \sqrt{\frac{(2l + 1) (l - m)!}{4\pi (l + m)!}} P_l^m(\cos \theta)e^{im\phi}.
   \]

   (a) Plot \( Y_0^0(\theta, \phi) \).
   (b) Plot \( Y_1^0(\theta, \phi) \).
   (c) Plot the magnitude of \( Y_1^1(\theta, \phi) \) and \( Y_1^{-1}(\theta, \phi) \).
   (d) Plot the magnitude of the real and imaginary parts of \( Y_1^1(\theta, \phi) \) and \( Y_1^{-1}(\theta, \phi) \).
   (e) Show that you can get a real basis by defining
   \[
   S_l^m(\theta, \phi) = \frac{1}{\sqrt{2}}[Y_l^m(\theta, \phi) + (-1)^mY_l^{-m}(\theta, \phi)] \quad \text{and} \quad S_l^{-m}(\theta, \phi) = \frac{1}{i\sqrt{2}}[Y_l^m(\theta, \phi) - (-1)^mY_l^{-m}(\theta, \phi)]
   \]
   (f) Plot the magnitude of \( S_1^1 \) and \( S_1^{-1} \).

4. (10 points) By combining problems 1 and 2, hopefully you have the tools to solve problems using spherical harmonics. Now let’s solve a simpler problem where we can assume that our solution to Laplace’s equation has no angular dependence (only dependence on \( r \)). Assume that I have a spherical metal shell of radius \( R \) which is held at a voltage \( V_0 \). Assuming that the potential at \( r = \infty \) is zero, what is the potential inside and outside of the sphere? Hint - you can use what you did on problems 1 and 2 as a foundation for this problem. But it’s really easier to start from scratch.