1. (8 points) The Legendre polynomials of the first kind, $P_n(x)$, can be generated in Mathematica using the function `LegendreP[n,x]`.
   
   (a) Plot the Legendre polynomials $P_0(x)$, $P_2(x)$, and $P_4(x)$ from $x = -1$ to 1 on a single graph. Label which line is which.
   
   (b) Plot the Legendre polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ from $x = -1$ to 1 on a single graph. Label which line is which.
   
   (c) You should have noticed a trend. Assuming that this trend continues (it does), what can you say about the value of $P_n(x)$ at $x = -1$ and $x = 1$?
   
   (d) What can you say about the number of times $P_n(x)$ crosses zero in the range $-1 \leq x \leq 1$?
   
   (e) What can you say about the value of $P_n(0)$ is $n$ is odd?

2. (4 points) Use Mathematica or look on the web to find the form of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$. Assuming that the trend continues (it does), what would be the highest power of $x$ that appeared in $P_n(x)$?

3. (8 points) The Legendre polynomials of the second kind, $Q_n(x)$, can be generated in Mathematica using the function `LegendreQ[n,x]`.
   
   (a) Plot the Legendre polynomials $Q_0(x)$, $Q_2(x)$, and $Q_4(x)$ from $x = -1$ to 1 on a single graph. Label which line is which.
   
   (b) Plot the Legendre polynomials $Q_1(x)$, $Q_3(x)$, and $Q_5(x)$ from $x = -1$ to 1 on a single graph. Label which line is which.
   
   (c) You should have noticed a trend. Assuming that this trend continues (it does), what can you say about the value of $Q_n(x)$ at $x = -1$ and $x = 1$?
   
   (d) What can you say about the number of times $Q_n(x)$ crosses zero in the range $-1 \leq x \leq 1$?
   
   (e) What can you say about the value of $Q_n(0)$ is $n$ is even?

4. (6 points) OK, I don’t have a story for this problem. Consider the following ODE/BC problem to find $Z(r)$ on the interval $0 \leq r \leq 1$.

   $$(1 - r^2)Z'' - 2r Z' + 6Z = 0$$

   $$Z(-1) = 1$$

   $$Z(r) = \text{finite}$$

   (a) Find the general solution to this equation.
   
   (b) Apply the boundary conditions to find $Z(r)$.

5. (4 points) Now let’s consider the more general associated Legendre’s equation. Solve this problem to find $y(x)$ on the interval $-1 \leq r < 1$.

   $$(1 - x^2)y'' - 2xy' + \frac{y}{x + 1} \left( \frac{12x^2 - 11}{x - 1} \right) = 0$$

   $$y(0) = 2$$

   $$y(x) = \text{finite}$$

   Feel free to use Mathematica to help you manipulate the associated Legendre polynomials. Note that in Mathematica $P_n^m(x)$ is written `LegendreP[n,m,x]`.

6. (10 points) Let’s use separation of variables to separate Laplace’s equation in spherical coordinates assuming that there is no azimuthal dependence (i.e. there is a symmetry such that $\phi$ won’t show up in our equation).
   
   (a) Write down Laplace’s equation in spherical coordinates
(b) Simplify the equation by assuming no azimuthal dependence.
(c) Use separation of variables to separate the equation into an equation for $r$ and an equation for $\theta$.
(d) Solve the $\theta$ equation by making the transformation $s = \cos(\theta)$ and making this look like Legendre’s differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$ 

(e) Note that the $r$ equation has the form

$$Ax^2y'' + Bxy' + Cy = 0.$$ 

This is a form of Euler’s differential equation, whose solutions, we reasoned, ought to look like $y = Dx^\nu$ where $D$ and $\nu$ are constants. Find the solutions to the $r$ part of the equation.
(f) Write the general series solution to Laplace’s equation for the case of axial symmetry.