1. (10 points) What do Bessel functions look like? To get a feel for it, we’ll consider some asymptotic limits and make some plots. First of all, note that in the limit of large $r$,

$$J_n(r) \approx \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{1}{2} n\pi - \frac{\pi}{4} \right),$$

and

$$Y_n(r) \approx \sqrt{\frac{2}{\pi r}} \sin \left( r - \frac{1}{2} n\pi - \frac{\pi}{4} \right).$$

So $J_n(r)$ is a constant times $1/\sqrt{r}$ times the cosine of $r$ with a phase shift. So it oscillates like a cosine, but damps out as $1/\sqrt{r}$. The function $Y_n(r)$ does exactly the same thing, except with a sine. In the limit of small $r$, $J_n(r)$ approaches

$$J_n(r) \approx \frac{r^n}{2^n \Gamma(1 + n)}.$$

Note that if $n$ is an integer, we could replace $\Gamma(1 + n)$ with $n!$. So if $n > 0$ it drops to zero as $r \to 0$ like $r^n$ times one over a bunch of stuff that grows rapidly with $n$. If $n = 0$, it approaches 1 as $r \to 0$. We won’t go into what $Y_n(r)$ does at small $r$, except to say that it becomes infinite at $r = 0$.

(a) Plot $J_0(r)$ and the two asymptotic limits I’ve given above from $r = 0$ to $r = 25$. Put all three functions on the same plot so that you can see how they match up.

(b) Now zoom in and plot the same things from $r = 0$ to 1. Print out the plot and calculate how small $r$ needs to be for our small $r$ approximation is within 1% of $J_0(r)$. To do that, remember that if I call my approximation $A_0(r)$, the fractional error is just $(A_0(r) - J_0(r))/J_0(r)$. So I just need to take the absolute value of that, set it equal to 0.01, and use FindRoot[ ] in Mathematica.

(c) Now plot $J_1(r)$ and the two asymptotic limits from $r = 0$ to 25.

(d) OK, enough with asymptotic limits, let’s just get a feel for what Bessel functions look like. On a single graph, plot $J_0(r)$, $J_1(r)$, $J_2(r)$ and $J_3(r)$. Label which function is which.

(e) Plot $Y_0(r)$, $Y_1(r)$, $Y_2(r)$, and $Y_3(r)$ from $r = 0$ to 25. Since these blow up at $r = 0$, limit the $y$ axis to go from -0.5 to 0.5.

2. (5 points) There’s a slight variation of Bessel’s equation known as the “parametric form of Bessel’s equation.” We’ve already seen it in homework problems, but let me formally introduce you:

$$x^2 y'' + xy' + (\lambda^2 x^2 - m^2)y = 0$$

where $m$ and $\lambda$ are constants. To solve this...

(a) First define $s = \lambda x$ and transform the equation into an equation for $y(s)$.

(b) This transformed equation should look familiar. Write down the general solution for $y(s)$, then transform back to get $y(x)$.

3. (10 points) There’s another equation very similar to Bessel’s equation which shows up a lot. It’s known as the modified Bessel equation:

$$r^2 R'' + r R' - (r^2 + p^2)R = 0.$$

Let’s solve it by transforming it into the regular, everyday, vanilla Bessel equation. To do this, we have to change the sign of the $r^2$. One thing that might work is to define $s = ir$ and write the equation in terms of $s$. The $i$ in this definition can make minus signs when it’s squared.

(a) Show that when you do this transformation, the equation you get for $R(s)$ is Bessel’s equation.

(b) The solution to your new equation is $AJ_p(s) + BY_p(s)$. Substitute $s = ir$ into this solution to find $R(r)$. 

(c) Show that the solution can be written in terms of the modified Bessel functions

\[ I_p(r) = i^{-p} J_p(ir), \text{ and} \]

\[ K_p(r) = \frac{\pi}{2} r^{p+1} [J_p(ir) + iY_p(ir)]. \]

Hint - first define the constant

\[ C \equiv B 2^{2i^{-p-2}}, \]

then try to make part of your solution look like \( K_p(r) \) times a constant. Then make what is left look like \( I_p(r) \) times a constant.

4. (15 points) Laplace’s equation \( \nabla^2 \phi = 0 \) describes the electrostatic potential in a region free of charge. We’ve already solved problems using this equation. One thing I’m working on in my research is the search for a possible non-zero photon rest mass. It turns that if photons have a tiny bit of rest mass, the accepted theory of electromagnetism is wrong. In one possible extension of electromagnetism that allows for a finite photon rest mass, instead of Laplace’s equation you get the equation

\[ \nabla^2 \phi - \gamma^2 \phi = 0 \]

where \( \gamma \) is a small parameter proportional to the photon rest mass. Note that this is just the Helmholtz equation, which we solved before using eigenfunction expansion. But we’re going to solve it by separation of variables this time because it gives us a nicer solution.

(a) Suppose that you are crazy enough to build an ion interferometer to search for electric fields inside of a conductor. Gauss’s law says there should be none, but effects of a possible non-zero photon rest mass might result in tiny fields. You’re probably going to have a long, skinny interferometer, so it would make sense to make conductors with a cylindrical geometry. Assume that the cylinder is long enough that we can ignore any \( z \) dependence and just use polar coordinates. Use separation of variables to find the general solution to this equation. Note that as you do this, one of your equations will look like the modified Bessel equation we saw above which has solutions \( I_p \) and \( K_p \).

(b) Now assume that our cylindrical tube has a radius \( a \) and is held at a potential \( V_0 \). What are the four boundary conditions for this problem?

(c) Apply the boundary conditions to find \( \phi(r, \theta) \). Note that by looking at the definitions of \( I_p \) and \( K_p \) in terms of \( J_p \) and \( Y_p \) you should be able to determine what each of them becomes in the limit as \( r \to 0 \). Also note that in classical electromagnetism, we would have just gotten that \( \phi \) was a constant inside the tube. But if photons have a tiny rest mass, we get something that changes very slightly with \( r \).

(d) Let \( a = V_0 = \gamma = 1 \) and plot \( \phi(r, 0) \) from \( r = 0 \) to \( a \).