1. (6 points) Let \( u = f(\vec{r})g(t) \) and show that you can separate the following equations into the Helmholtz equation for \( f \) and another ODE for \( g \):

(a) The N-dimensional wave equation

\[
c^2 \nabla^2 u = u_{tt}
\]

(b) The N-dimensional diffusion equation

\[
k \nabla^2 u = \rho \mu_t
\]

2. (15 points) Since the Helmholtz equation basically says “The Laplacian acting on \( u \) is equal to a constant times \( u \),” the solutions to the Helmholtz equation are eigenfunctions of the Laplacian. When we solved the wave equation in polar coordinates, we got a solution to the spatial part that looked like

\[
f(r, \theta) = [A_\mu \gamma J_\mu(\gamma r) + B_\mu \gamma Y_\mu(\gamma r)][C_\mu \cos(\mu \theta) + D_\mu \gamma \sin(\mu \theta)].
\]

These are the eigenfunctions of the Laplacian in polar coordinates. Let’s not assume anything about boundary conditions in this problem (so let’s not throw out any terms or put any limits on what \( \mu \) or \( \gamma \) can be. The definition of an eigenfunction of an operator is that it satisfies the equation

\[
\hat{D} f = \lambda f,
\]

where \( \hat{D} \) is the operator (which is operating on the function \( f \)), and \( \lambda \) is a constant known as the eigenvalue of the eigenfunction. What are the eigenvalues of your eigenfunctions? Remember that Bessel’s equation tells us that if \( y \) is a Bessel function of order \( \mu \), then

\[
x^2 y'' + xy' + (x^2 - \mu^2)y = 0.
\]

Or, taking things to the other side of the equation, dividing by \( x^2 \), and plugging in \( J_\mu(x) \) for \( y \), we get

\[
\frac{\partial^2}{\partial x^2} J_\mu(x) + \frac{1}{x} \frac{\partial}{\partial x} J_\mu(x) = \left( \frac{\mu}{x^2} - 1 \right) J_\mu(x).
\]

Or, alternatively, we could have plugged in \( Y_\mu(x) \) as well. Also note that the way I’ve defined the eigenvalue differs from the book’s definition by a minus sign (the book absorbed the minus sign in your answer into the definition). I believe my definition is more conventional. It’s the one that Wikipedia uses, anyway.

3. (15 points) Consider a long cylindrical metal tube with inner diameter \( R \) filled with charge with a constant density \( \rho_0 \). We can find the potential anywhere inside the cylinder by solving Poisson’s equation:

\[
\nabla^2 \phi(r, \theta) = -\rho(r, \theta)/\epsilon_0.
\]

We live in three dimensions, but if the column of charge is infinitely tall, then \( u_{zz} = 0 \) and the equation in cylindrical coordinates turns into the same equation in polar coordinates.

(a) Assume that the potential of the tube is zero. Use this information along with the periodic boundary conditions we often use in \( \theta \) and the knowledge that the potential is finite everywhere to write the solution inside the cylinder, \( \phi(r, \theta) \), as a sum of eigenfunctions of the Laplacian which fit our boundary conditions.

(b) Insert the form of \( \phi \) that you wrote above into the PDE. Remember, you already found the eigenvalues of each eigenfunction in the last problem, so you shouldn’t have to re-calculate them.

(c) Now write the charge inside the cylinder, \( \rho_0 \), as a sum of the same eigenstates. Note that

\[
\int_0^{\alpha_0} x J_0(x) dx = \alpha_0 J_1(\alpha_0)
\]

where \( \alpha_0 \) is the \( n^{th} \) root of \( J_0(x) \).

(d) Plug that sum into the PDE and solve it to find \( \phi(r, \theta) \).
4. (4 points) We found on the last homework that if we solved Laplace’s equation with the boundary conditions $\phi(R, \theta) = Q \sin(q\theta)$, $\phi(r, \theta) = \text{finite}$, $\phi(r, \theta) = \phi(r, \theta + 2\pi)$, and $\phi_r(r, \theta) = \phi_r(r, \theta + 2\pi)$, where $q$ is an integer, we get

$$\phi(r, \theta) = \frac{Q}{R^q} \frac{r^q}{q} \sin(q\theta).$$

Use this information and the principle of superposition to find what the potential will be inside of a long cylinder of charge of radius $R$ with a potential at the edge of the charge given by $\phi(R, \theta) = Q \sin(q\theta)$. 