1. (18 points) Start with Laplace’s equation in polar coordinates.
   
   (a) Use separation of variables to separate the PDE into two ODEs.
   
   (b) Find the solution to the $\theta$ equation.
   
   (c) The $r$ equation is solved differently if the separation constant is zero. So first consider that case and find the solution to the $r$ equation when the separation constant is zero.
   
   (d) Now let’s see what we get when the separation constant is not zero. There are different ways to solve an equation like this. The approach I want you to use is a technique I call “being clever.” Note that the second derivative term is multiplied by $r^2$, the first derivative term is multiplied by $r$, and the “zeroth” derivative term is not multiplied by $r$. If the solution was just a constant times $r$ to some power, the second derivative term would drop the power of $r$ by two, which is exactly compensated for by the $r^2$, etc. So, being clever, let’s assume that the solution is equal to a constant times $r^n$, plug it in, and see what $n$ needs to be. Hint - you should find two possible values for $n$, so your final solution should look have two terms - it should look like $Ar^{\text{some power}} + Br^{\text{some power}}$.
   
   (e) Put the two solutions together to get the general solution to the PDE. Note that your general solution should involve a sum over all possible values of your separation constant except zero plus something outside of the sum that deals with the possibility that the separation constant is zero.

2. (10 points) I take a wire and stretch it into a “wavy circle” and dip it into bubble solution to make a soap film stretched across the wire loop. The wire is shaped in a way such that if you looked down on it from above it would look like a circle of radius $R$. But looked at from an angle, you realize that the side waves up and down sinusoidally such that the edge of the soap film obeys the equation $u(R, \theta, t) = Q \sin(q\theta)$
   
   where $Q$ and $q$ are constants, $q$ being an integer, as shown in the figure below.
   
   (a) The shape of the soap bubble is given by Laplace’s equation, which is second order in $r$ and $\theta$. This means we should have two boundary conditions in each. What is the other $r$ boundary condition, and what are the two boundary conditions in $\theta$?
   
   (b) Apply the boundary conditions to the general solution you found in problem one to find $u(R, \theta)$, the shape of the soap film in equilibrium (i.e. when you are holding very still and the air near the film is not moving).
   
   (c) Let $Q = R = 1$ and $q = 5$ and make a plot of $u(r, \theta)$. If you use Mathematica you can do this by noting that $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ and using Plot3D. Note that Mathematica’s arctan function will accept two arguments - [x,y] instead of [y/x] - on order to remove the $\pi$ ambiguity in arctan. Also note that your solution has no meaning for $r > R$. You can use the “RegionFunction” option to keep Mathematica from plotting anything outside of the relevant area. For example, if you define the function u[r, t, theta], you could plot it using the command
   
   Plot3D[u[r, Sqrt[x^2 + y^2]], normals [x, y], [x, -R, R], [y, -R, R], RegionFunction -> Function[x, y, x^2 + y^2 < R^2], PlotRange -> All]

3. (7 points) Imagine a very long coaxial cable made of a very long cylindrical conductor of radius $a$ centered inside a long hollow conducting cylinder with an inner radius $b$. Let’s assume that the space in between is empty - vacuum. The electrostatic potential $\phi$ between the two conductors (i.e. $a \leq r \leq b$) is given by Laplace’s equation. Since the conductors are very long, we can use the polar form. Find $\phi(r, \theta)$ in this region assuming that the inner conductor is held at a potential $V_a$ and the outer conductor is held at a potential $V_b$. In order to avoid putting units into the natural log, write that part of your solution as $\ln(r/b)$ rather than just $\ln(r)$. This can be completely justified by just redefining some constants and will make your solution make more sense. Finally, assume $a = 1$, $b = 2$, $V_a = 1$, and $V_b = 0$ and plot $u(r, \theta)$. The plot function I used is...
4. (5 points) When we solve electrostatic problems, we often assume that the potential at $r = \infty$ is zero. But in the case of a problem with no $z$ dependence, our conductor goes all the way to $\infty$ in the $z$ direction. Assume that I have an infinitely long conducting cylinder of radius $R$ held at a potential $V_0$. With no assumptions about what the potential is at infinity other than the fact that it is finite, find $\phi(r, \theta)$ outside of the conductor. Again, to avoid units problems, let’s write the natural log part as $\ln(r/R)$ rather than $\ln(r)$.