1. (15 points) Bessel functions of the first kind of a given order make a complete orthogonal set over certain regions. But the orthogonality integral is a little different from the sine/cosine orthogonality integral. Instead, the integral you get is

$$\int_0^1 r J_m(\alpha_{mn} r) J_m(\alpha_{ml} r) dr = \frac{\delta_{nl}}{2} [J_{m+1}(\alpha_{mn})]^2$$

where $m$, $n$, and $l$ are integers and $\delta_{nl}$ is the Kronecker delta functions, which is equal to 1 if $n = l$ and zero if not. Imagine I have a function $f(r)$ defined in the region $0 \leq r \leq a$. I want to write that function as a sum of Bessel functions of the first kind of order $m$:

$$f(r) = \sum_{n=1}^{\infty} A_n J_m(\alpha_{mn} r/a).$$

Use the integral above to find a formula for $A_n$.

2. (25 points) Consider a circular drum head of radius $a$. The circumference of the drum head is fixed. The time evolution of the drum head is described by the 2D wave equation which, in polar coordinates, looks like

$$u_{tt} = c^2 \left( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} \right).$$

Imagine that at time $t < 0$ the drum head is motionless. Then I hit it with a small drumstick such that

$$u(r, \theta, 0) = 0, \quad u_t(r, \theta, 0) = \Gamma \delta(r - r_h) \delta(\theta - \theta_h).$$

where $\Gamma$, $r_h$, and $\theta_h$ are constants.

(a) Write down the boundary conditions for this problem.

(b) Use separation of variables to turn the PDE into a set of three ODEs.

(c) Let’s find the general solution to the $r$ equation by noting that it looks kind of like Bessel’s equation:

$$x^2 y'' + xy' + (x^2 - n^2) y = 0$$

which has the solution

$$y = AJ_m(x) + BY_m(x).$$

To do this, define $q \equiv \lambda r$ and transform your equation into this new coordinate system. Then it should look like Bessel’s equation whose solution is known. Finally, transform your solution back into the $r$ representation.

(d) Solve the other two equations and find the general solution $u(r, \theta, t)$.

(e) Apply your boundary conditions to find a series solution to the problem with unknown constants.

(f) If I bang on a round drum head with radius $a$, what frequencies could I potentially hear in the sound? Note that I asked for frequency, not angular frequency. Remember to state what values the integers in the answer can have.

(g) Apply your initial conditions to find $u(r, \theta, t)$ with no unknown constants. I’m not going to make you plot this one because it can take some time. But if you decide to do it on your own, here are some hints - In Mathematica, $J_m(x) = \text{BesselJ}[m, x]$, and the $n^{th}$ root of $J_m$ is $\text{BesselJZero}[m, n]$. It takes Mathematica some time to calculate Bessel functions, and even more to find zeros. So make a table of the zeros, and then call the table in your sum rather than making it calculate over and over in each sum at each point. You can speed things up even more by making a table with your constants for each term as well. Also, if you want it to plot faster, instead of making a 3D plot, let $\theta_h = 0$ so that the drum strike is on the $x$ axis, and just plot a slice at $y = 0$. You can do this by letting $r = |x|$ and $\theta = (1 - x/|x|)(\pi/2)$. If you do take the time to make a 3D plot, remember that your solution is only valid when $r < a$. You can tell Mathematica to only plot the valid region by adding the statement RegionFunction -> Function[{{x, y}, x^2 + y^2 < a^2}] to your plot command.