1. (10 points) The textbook (Asmar) gave you the d’Alembert solution to the wave equation without proof. It also presented it as a solution for a wave on a finite medium which is fixed at both ends \((u(0, t) = u(L, t) = 0)\). But the d’Alembert solution is more general than that, and it is easily derived when we use it to describe waves on an infinite string. So now consider an infinitely long string (or one at least long enough that any disturbance we make in the middle area of it won’t have time to reach the boundaries during our experiment so that we can treat it as if it were infinite). We know that any wave of the form \(u(x, t) = f(x - ct)\) or \(u(x, t) = f(x + ct)\) is a solution to the wave equation. For lack of time, I won’t make you prove, but will let you accept as fact that any solution to the wave equation can be written as \(u(x, t) = B(x + ct) + C(x - ct)\), where \(B\) and \(C\) are functions of one variable. Now let’s derive d’Alembert’s solution from that

(a) Let \(f(x) = u(x, 0)\) and \(g(x) = u_t(x, 0)\). Apply the \(f(x)\) initial condition to the solution \(u(x, t) = B(x + ct) + C(x - ct)\) to get an equation which relates \(f(x)\) to \(C(x)\) and \(B(x)\).

(b) Now apply the \(g(x)\) initial condition. To do this, you will need to use the chain rule on \(C(x - ct)\) and \(B(x + ct)\). And then, only after evaluating the derivative, you will set \(t = 0\). In order to convert from \(C'(x)\) and \(B'(x)\), you should integrate both sides of the equation from \(x_0\) to \(x\). This should give you another equation that relates \(C(x)\) and \(B(x)\) to \(\int_{x_0}^x g(x)dx\).

(c) Now solve the system of two equations to find \(C(x)\) and \(B(x)\), and then plug them back into your solution to get the D’Alembert solution. Remember that

\[
\int_a^b f(x)dx = -\int_b^a f(x)dx,
\]

and

\[
\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.
\]

2. (6 points) Not every point in the initial conditions will affect every point on the string at all times. For example, if I made a pulse in the middle of an infinite string, a long distance from the pulse’s starting point the string will just sit there for a long time, ignorant of the pulse. Then the pulse will travel by, and then the string will just sit there again.

(a) Imagine that I have an infinite string with the initial conditions \(u(x, 0) = f(x)\) and \(u_t(x, 0) = g(x)\). At some time \(t_{look}\) I look at some position \(x_{look}\). What parts of \(f(x)\) will influence the value at \(x_{look}\) at time \(t_{look}\)? (In other words, what range of values of \(x\) in \(f(x)\) will have an influence on this point.)

(b) What parts of \(g(x)\) will influence the value at \(x_{look}\) at time \(t_{look}\)?

3. (9 points) Imagine a very long water pipe - long enough to treat as if it’s infinite for this problem. Let’s assume that as sound waves propagate through the water, the displacement of water inside the pipe follows the wave equation \(s_{tt} = c^2 s_{xx}\). I tap on the side of the pipe, creating a disturbance at time \(t = 0\) of the form

\[
s(x, 0) = Ae^{-\alpha x^2}
\]

\[
s_t(x, 0) = 0
\]

where \(A\) and \(\alpha\) are constants.

(a) What is \(s(x, t)\)?

(b) Let \(A = \alpha = c = 1\) and make a plots of \(s(x, t)\) from \(x = -5\) to \(5\) at times \(t = 0, 1, 2,\) and \(3\).

4. (9 points) Now imagine that I give the water in the pipe a disturbance of the form

\[
s(x, 0) = 0
\]

\[
s_t(x, 0) = \frac{A}{x^2 + 1}
\]

where \(A\) is a constant. Note that

\[
\int \frac{1}{x^2 + 1}dx = \arctan(x).
\]
(a) Find an equation for $s(x, t)$.

(b) Let $A = c = 1$ and make plots of $s(x, t)$ from $x = −100$ to $100$ at times $t = 0, 5, 10, \text{ and } 50$.

5. (6 points) We can also apply this solution to waves on a finite string with fixed ends by imagining the string to be a finite piece of an infinite string whose initial conditions are the initial conditions of the string reflected about the origin and then repeated over and over again to make an odd periodic function which repeats with a period of $2L$. Imagine that I have a guitar string of length $L$ which is described by the equation $u_{tt} = c^2 u_{xx}$ and has the boundary conditions $u(0, t) = u(L, t) = 0$. The string is not moving at time $t = 0$ but has an initial shape of $u(x, 0) = 4 \sin(\pi x / L) + \sin(4 \pi x / L)$. Use the d’Alembert solution to quickly find $u(x, t)$.

If you want, for your own enjoyment but not to be graded, you can plot your solution at various times. Then solve the same problem by separation of variables. You’ll get a solution that looks different - but will generate the same plots.

For your own enlightenment (not required, not graded) We derived the d’Alembert solution above starting with the idea that any solution to the wave equation could be written in the form $u(x, t) = B(x + ct) + C(x − ct)$, where $B$ and $C$ are functions of one variable. In this problem we’ll make a transformation which will allow us to solve the wave equation $u_{tt} = c^2 u_{xx}$ in such a way that our solution is in this form.

(a) We know from experience that any wave of the form $y(x − ct)$ will satisfy the wave equation. Likewise, we know that $y(x + ct)$ will be a solution, because if we change $c$ to $−c$, it doesn’t change the wave equation. Let’s transform into coordinates which travel with these special waves. Let’s define $\eta \equiv x + ct$ and $\xi \equiv x − ct$ - this is known as the “canonical transformation.” Using the chain rule I can write $u_t$ as

$$u_t = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} = u_\eta \eta_t + u_\xi \xi_t = cu_\eta − cu_\xi.\$$

Do something similar to write $u_x$ in terms of $\eta$ and $\xi$.

(b) Now write $u_{tt}$ and $u_{xx}$ in terms of $\eta$ and $\xi$. Note that for single-valued reasonable functions (ones that describe everyday physical things - black holes not included) $u_{ab} = u_{ba}$.

(c) Plug what you found in part (b) into the wave equation to get a new wave equation written in terms of $\eta$ and $\xi$. Note that, if you do everything right, it should reduce down to one second-order derivative on one side of the equation, and zero on the other.

(d) Integrate both sides of the equation with respect to $\eta$, and then with respect to $\xi$, to get the solution. Note that the “constants” that you get when you do the indefinite integrals are only constant with respect to the integration variable. In other words, they are a function of the other variable. You should get something that looks like

$$u = \int A(\xi) d\eta + B(\eta).\$$

But notice that the integral part only depends on $\xi$, and $A$ can be any function. This means that the integral of $A$ can be any function and still satisfy the PDE. Which means that the integral can integrate to be any function of $\xi$. So why don’t we just call that whole integral $C(\xi)$.

(e) Transform your solution back to $x$ and $t$ coordinates. We have now proven that by adding a function of $x + ct$ to a function of $x − ct$, we get a solution to the wave equation. Didn’t we already know that? But now we’ve shown that it is a general form of solution - that we can write any solution in this general form.