1. (10 points) OK, we finally have the tools to solve this problem we’ve been dying to solve. We wanted to be able to calculate the temperature along a transistor lead as we soldered it. A few assignments ago we derived the 1-D diffusion equation and found a general solution for it. But we couldn’t fit the boundary conditions to our solution. Then on the last assignment we did a transformation so that we could get a solution that fit our boundary conditions. When you did this, you should have gotten

\[ T = T_H + \sum_{n=0}^{\infty} B_n \cos \left( \left( n + \frac{1}{2} \right) \frac{\pi}{L} x \right) e^{-\frac{(n+1/2)^2 \pi^2 x^2}{4L^2}}. \]

But then we couldn’t fit this solution to our initial conditions (that \( T(x, 0) = T_0 \)). Now we know how to do this - all you need to know is that if \( n \) and \( m \) are integers, then

\[ \int_0^1 \cos \left( (n + 1/2) \pi x \right) \cos \left( (m + 1/2) \pi x \right) dx = 0 \]

unless \( n = m \) (in which case you should be able to solve the integral). In other words, the terms in the sum part of our solution are orthogonal in \( x \) with respect to the weight function \( r(x) = 1 \).

(a) Use this information to find \( T(x, t) \).

(b) Use a computer to plot \( T(x) \) at times \( t = 0.001, 0.01, 0.1, 1, \) and 10. To make this possible, let \( k/L^2\rho c = 1 \), and let \( T_0 = 1 \) and \( T_H = 2 \), and only sum up the first 50 terms.

2. (14 points) We finally got the answer we’ve been after all this time! But in the process, we’ve only considered fixed temperature and zero heat flow boundary conditions. So let’s do the same problem, only let’s assume that the transistor is a big heat sink so that we can have a “surrounding medium at a constant temperature” kind of boundary condition\(^2\). And let’s let the soldering iron not produce a constant temperature, but a constant (non-zero) rate of heat flow. To make things simpler, I’m going to drop most of the constants - making them either zero or one as appropriate. This gives us the following problem:

\[ \text{PDE :} \quad T_{xx} = T_t \quad 0 < x < 1 \]
\[ \text{BCs :} \quad T_x(0, t) - T(0, t) = 0 \]
\[ \quad T_x(1, t) = 1 \quad 0 < t < \infty \]
\[ \text{IC :} \quad T(x, 0) = x + 1 - \sin(q x) - q \cos(q x) \quad 0 < x < 1 \]

where \( q = 0.860334 \).

(a) Make sure that you know where every term comes from. Ask yourself why is there a \( T_{xx} \), a \( T \), and a \( T_t \) in the PDE? Ask yourself what, physical meaning the boundary conditions have, etc. If you’re confused, get help from me, a fellow student, or anyone who you trust about this. If you can’t get help right now, go on to the next part, but be sure to get help as soon as you can. Then write “Yes, I understand what everything physically means in this problem.”

(b) We need homogeneous boundary conditions. So we need to write \( T \) as \( f(x, t) + g(x) \), where \( g(x) \) is some function that makes the boundary conditions for \( f(x, t) \) homogeneous. We’ve done this before, but now we have different boundary conditions... We could try guessing what to use for

\(^2\)As a reminder, remember that the heat flow at a location is proportional to \(-\partial T/\partial x\). If you are in a medium at a constant temperature \( T_{\text{medium}} \), then the rate that heat flows into you is proportional to \( T_{\text{medium}} - T \). So, for a boundary on the left, this boundary condition looks like \( \partial T/\partial x = h(T - T_{\text{medium}}) \), where \( h \) is a constant. On the right hand side, \( T_{\text{medium}} - T \) is proportional to the heat flowing in the other direction into the lead. So in this case \( \partial T/\partial x = h(T_{\text{medium}} - T) \).
We know that we want something that doesn’t change the B.C. at \( x = 0 \) (which is already homogeneous), so \( g_x(0) \) must equal \( g(0) \). We also want to homogenize the B.C. at \( x = 1 \), so we want \( g_x(1) = 1 \). But there are many, many functions that would satisfy this criterion. Try \( x^2/2 \) and show that, while this does homogenize the B.C.s, it makes the PDE inhomogeneous.

(c) A really good guess for the \( g(x) \) we should use is the solution for \( T \) at \( t = \infty \) - the steady state solution for \( T \). Find that steady state solution and use it to transform our equation for \( T \) into an equation for \( f \). Write down the new PDE and the new B.C.s.

(d) Now, noting that 0.860334 is, to 6 significant figures, a solution to the transcendental equation \( \alpha \tan \alpha = 1 \), find \( T(x, t) \).

(e) Plot \( T(x, t) \) as a function of \( x \) from \( x = 0 \) to 1 at times \( t = 0, 2, 4, \) and 1000.

(f) Do the plots make sense to you? Don’t answer this question until the answer is yes (you will not receive credit for “no”).

3. (16 points) Now we’ll explore what happens when we let heat flow out of the side of the lead into the air. Remember that the equation we started with looked like

\[
kT_{xx} = \rho c T_t,
\]

in other words, the rate of change of the temperature at some location depends on the the curvature of \( u \) in position. But if heat flows out the side, that will induce a negative change in the temperature of the form

\[
\mu(T - T_0) = a T_t,
\]

where \( \mu \) is a constant. Putting them together, we get

\[
kT_{xx} - \mu(T - T_0) = T_t.
\]

To make life simpler, I’ll set constants to 1 or zero and give you homogeneous boundary conditions.

\[
\begin{align*}
PDE : \quad T_{xx} - T &= T_t \quad 0 < x < 1 \\
BCs : \quad T(0, t) &= 0 \\
&\quad T(1, t) = 0 \quad 0 < t < \infty \\
IC : \quad u(x, 0) &= 1 \quad 0 < x < 1
\end{align*}
\]

(a) Write \( T(x, t) \) as some function \( w(x, t) \) times a function of just time such that the differential equation becomes the simple 1-D diffusion equation (with no term representing heat flow out of the side). Hint: the other function is just the solution to the problem if heat is not allowed to flow in the \( x \) direction - i.e., if each piece of the medium was isolated from each other.

(b) Solve the following ICBC problem (i.e., find \( T(x, t) \)). Note that

\[
\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = 0
\]

unless \( n = m \). And note that you can simplify things by thinking about odd \( n \) and even \( n \) terms separately.

(c) Plot the sum of the first 50 non-zero terms in \( T(x, t) \) as a function of \( x \) for \( t = 0.001, 0.01, 0.1, \) and 1.