1. (9 points) State whether the following boundary conditions are Dirichlet (1st type), Neumann (2nd type), or Robin (3rd type) boundary conditions.
   
   (a) \( u(0, y, t) = 0 \).
   
   (b) \( u(0, y, t) + u_x(0, y, t) = 0 \).
   
   (c) \( u_x(L, y, t) = \gamma \).
   
   (d) The end of a rod is held at a constant temperature \( T_a \).
   
   (e) The end of a rod is held in ice water.
   
   (f) The end of a rod is connected to a heater which injects a constant power \( P_a \) into the rod.
   
   (g) A guitar string is held motionless at one end.
   
   (h) A column of air in an organ pipe is open to the air such that the pressure at that end is always \( P_0 \).
   
   (i) The end of a guitar string is connected to a spring with spring constant \( k \).

2. (6 points) Show that the following equations can be separated into an ODE that only depends on time, and the Helmholtz equation.
   
   (a) The diffusion equation: \( u_t = \gamma \nabla^2 u \).
   
   (b) The wave equation: \( u_{tt} = c^2 \nabla^2 u \).

3. (15 points) A log with a radius \( R \) and insulated ends is caught in a river such that half of the log is under water, and half is in the air. The water is moving quickly, and it is a very windy day. Because of this, half of the log's surface remains at the temperature of the water \( T_w \), and half remains at the temperature of the air \( T_a \):
   
   \[ T(R, \theta) = T_a H(\pi - \theta) + T_w H(\theta - \pi) \text{ for } 0 \leq r \leq R, \ 0 \leq \theta < 2\pi. \]

   (a) What is the PDE that we solve to get the steady-state temperature \( T(r, \theta) \)?
   
   (b) Assume that the insulation on the ends of the log is good enough that there is no heat flow (and therefore no temperature gradients) in the z direction. Then use separation of variables to turn the PDE into an ODE for the \( r \) part and an ODE for the \( \theta \) part.
   
   (c) Solve the ODE for the \( \theta \) dependent part of the solution. Then apply the boundary condition that \( T(r, \theta) = T(r, \theta \pm 2\pi) \).
   
   (d) The book calls the equation that you get for the \( r \) part “Euler’s equation” and gives you the solutions. But instead of just writing them down, let’s think a bit about the Euler-Cauchy equation. This equation has the form
   
   \[ xy_{xx} + bxy_x + cy = 0, \]

   where \( b \) and \( c \) are constants. The problem with this problem is the fact that we have \( x \)’s in the equation making the coefficients not constant. What we want to do is make a transformation from \( x \) to some other variable \( s \) that gets rid of the non-constant coefficients, solve the problem, and then transform back. Use the chain rule to write the equation above in terms of \( x \), \( y \), \( y_s \), \( y_{ss} \), \( s_x \), \( s_{xx} \), \( b \), \( c \), and \( x \).
(e) Notice that we could make all of the $x$'s and $s$'s disappear in the equation you found above if we found an $s$ such that $s_x = W/x$ and $s_{xx} = X/x$ where $W$ and $X$ are constants. Let $s_x = 1/x$, and find what $s$ is in terms of $x$ and arbitrary constants. Then find $s_{xx}$ in terms of $x$ and arbitrary constants.

(f) Let’s not solve the general form of the Euler-Cauchy equation, but instead let’s take on the simpler problem of solving our specific ODE for the radial part of $T(r, \theta)$. Make the substitution, solve the ODE, and then transform back to find the radial part of the solution. Remember that, since this is a second order ODE that should have two solutions. If there is a circumstance where your two solutions are the same, you need to find a second solution for this second special case.

(g) Apply the boundary condition at $r = R$ (given above) and the boundary condition at $r = 0$ (it doesn’t blow up) to find the steady-state solution $T(r, \theta)$.

(h) Set $R = 1$, $T_a = 1$ and $T_w = 0$ and make a surface plot of the temperature cross section of the log. For simplicity, only sum up to 50 terms in your solutions.

4. (5 points) A very long metal pipe, with a radius $R$, is sliced in half lengthwise. One half is held at a potential $V_1 = 0$. The other half is held at a potential $V_2 = 1$. Make a surface plot of a cross section of the potential in the pipe.

5. (5 points) Consider circle of radius $R$, and a point $(r, \theta)$ inside of a circle. Let $s(\alpha)$ be the distance from that point to a point on the circle which is an angle $\alpha$ from the $x$ axis, as shown below. Show that the Poisson Integral Formula is just the weighted average of the potential on the surface with a weight equal to $w(\alpha) = 1/s^2$ times a constant (with respect to the integration variable). (Note that the constant is there to normalize the weights. For example, if $g(\alpha)$ was the same for all angles $\alpha$, the weighted average should just be equal to $g(\alpha)$ at any $\alpha$. But it won’t be unless the integral of $w(\alpha)d\alpha$ is equal to 1.)