1. (10 points) Let’s try to solve the modified Bessel equation
\[ r^2 R'' + r R' - (r^2 + p^2)R = 0 \]
by transforming it into the regular, everyday, vanilla Bessel equation. To do this, we have to change the \( p^2 \) into a negative \( p^2 \). We could do this by making \( p \) be imaginary. But it’s not fair to go around changing constants. Instead, let’s transform the independent variable by defining \( s = ir \) and see where that gets us.

(a) Show that when you do this transformation, the equation you get for \( R(s) \) is Bessel’s equation.
(b) The solution to your new equation is \( A J_p(s) + BY_p(s) \). Substitute \( s = ir \) into this solution to find \( R(r) \).
(c) Show that the solution can be written in terms of the modified Bessel functions
\[
I_p(r) = i^{-p} J_p(ir), \quad \text{and} \quad K_p(r) = \frac{\pi i^{p+1}}{2} [J_p(ir) + iY_p(ir)].
\]

2. (15 points) The equation that describes the electrostatic potential in a region in which things aren’t changing in time is Laplace’s equation: \( \nabla^2 V = 0 \). We’ll take a look at this equation in the next unit. It turns out, however, if photons have a tiny bit of rest mass, you can create a modified version of electromagnetism in which this equation is replaced with
\[ \nabla^2 V - \gamma^2 V = 0 \]
where \( \gamma \) is a small parameter proportional to the photon rest mass.

(a) Suppose that you are crazy enough to build an ion interferometer to search for possible effects of a non-zero photon rest mass. You’re probably going to have a long, skinny interferometer, so it would make sense to make conductors with a cylindrical geometry. Use separation of variables to solve this equation in cylindrical coordinates to find a general solution for \( V(r, \theta, z) \) of the form \( V(r, \theta, z) = R(r) \Theta(\theta) Z(z) \).
(b) Now assume that the ions travel through a long, cylindrical tube. Since the conductor is really long, let’s pretend that it’s infinitely long. What does our series solution become if we assume that there is no \( z \) dependence?
(c) Now assume that our cylindrical tube has a radius \( a \) and is held at a potential \( V_0 \). What are the boundary conditions imposed by the tube, and what are the boundary conditions on \( \theta \)?
(d) Apply the boundary conditions to find \( V(r, \theta) \).

3. (15 points) Some day you may come across the classical problem of the lengthening pendulum. In this problem, you assume that you have a mass on a string oscillating back and forth. But over time, you increase the length of the string. I won’t go over how to set up this problem, but I’ll just give you the ODE/IC problem to solve below. If the angle that the string makes with respect to the vertical is denoted as \( \theta \), and the length of the string at any given time is \( l = l_0 + vt \), the ODE you get is
\[
\frac{dl^2 \theta}{dl^2} + 2 \frac{d\theta}{dl} + \frac{g}{l} \theta = 0
\]
where \( g \) is acceleration due to gravity. Notice that this equation gives \( \theta \) as a function of \( l \), not time.
(a) Find the solution \( \theta(l) \).

(b) Applying initial conditions is hard to do analytically. So let’s just get the character of the motion by assuming that we have an initial condition which makes the \( Y_n(r) \) term go to zero. Set the constant in the \( J_n(r) \) term to some arbitrary constant, and plot \( \theta(l) \) from \( l = 0.02 \) to \( 0.5 \) assuming \( g = 10 \) and \( v = 0.05 \).

(c) Note that as the pendulum oscillates, its \( x, y \) coordinates as a function of \( l \) are given by

\[
    x = l \sin(\theta) \quad \text{and} \quad y = -l \cos(\theta).
\]

Make a parametric plot that shows the path that the pendulum traces out as it oscillates from \( l = 0.02 \) to \( 0.5 \). Note that in Mathematica you can make a parametric plot by typing “ParametricPlot[{blah,booga},{l,0.02,0.5}]” where “blah” and “booga” are the functions that generate the \( x \) and \( y \) coordinates from \( l \). In MatLab you can make a parametric plot by simply making one vector \( x \) with the \( x \) values at each value of \( l \) and another vector \( y \) with the \( y \) values at each value of \( l \), and then using the command plot(x,y). Make sure that you take fine enough steps in \( l \) to capture the dynamics.