1. (20 points) Imagine a very thin, tall glass of water (so thin that we can treat this as a one-dimensional problem). The speed of sound in water is just $\sqrt{B/\rho}$, where $B$ is the bulk modulus of water and $\rho$ is the density of water. So the PDE that describes this situation is

$$s_{tt} = \frac{B}{\rho} s_{xx}$$

(a) The top of the glass is open to the air. Because of this, the force at $x = L$ will be equal to $P_0 A$ where $P_0$ is atmospheric pressure and $A$ is the cross sectional area of the water. Write down a boundary condition for this end. Remember, when we derived the wave equation for a similar system, we found that force was $F = -ks_pdx$, where $k$ is the “spring constant” for one of our infinitesimal slices, and we found that $k = BA/dx$.

(b) At the bottom, the water is pressed up against rigid glass. If we assume that the bulk modulus of the glass is much greater than the bulk modulus of the water, we can approximate that the water at $x = 0$ cannot move. Write down a boundary condition that describes this situation.

(c) Homogenize the boundary conditions by letting $s = u + A[1 - x/L] + B[x/L]$ where $A$ and $B$ are constants chosen to homogenize the BC. Then use separation of variables and apply the transformed BC to find a general series solution for $u(x, t)$ (i.e. an infinite sum with coefficients which will depend on the initial conditions).

(d) Imagine now that I bump the glass, creating the initial condition $s(x, 0) = -P_0 x/B$, $s_t(x, 0) = v_0 \delta(x - L/2)$. Transform these conditions into conditions for $u(x, 0)$ and $u_t(x, 0)$, and apply them to find $u(x, t)$. Then transform back to find $s(x, t)$. To do this, note the orthogonality relationship for $\sin((n + 1/2)\pi x/L)$:

$$\int_0^L \sin \left( \frac{(n + 1/2)\pi x}{L} \right) \sin \left( \frac{(m + 1/2)\pi x}{L} \right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{otherwise} \end{cases}$$

Also note that $\int x \sin(x)dx = \sin(x) - x \cos(x)$.

(e) Let $B = P_0 = \rho = L = v_0 = 1$ and plot your answer from $x = 0$ to 1 for times $t = 0.01, 0.2, 0.4, 0.6, 0.8,$ and 1.0.

2. (8 points) Now imagine that the bottom of the glass isn’t entirely solid. But if you compress it a little, it pushes back really hard. Let’s imagine that it behaves like a spring, such that if you push it down an amount $\epsilon$, it pushes back with a force $ke$. Write a boundary condition that describes this situation.

3. (12 points) A guitar string has a linear mass density $\mu$ and is tuned up to a tension $T$. At one end the string is fixed rigidly to the nut, such that it can’t vibrate at $x = 0$. At the $x = L$ end, the string passes over the bridge, which transmits vibrations from the string to the top plate. The top plate is slightly flexible so that it can move air and make the guitar loud. Let’s approximate the bridge/top plate with a spring with a spring constant $k$, as shown in the figure below. Assume that the spring is constrained so that it can only move up and down. Write down the boundary conditions for this problem.