1. (6 points) Use the ratio test, integral test, or alternating series test find the region of convergence for
the following series. State whether they are absolutely convergent or conditionally convergent in that
region, or if they are divergent everywhere. (Note - the ratio test fails along the edge of the region - for
this assignment don’t worry whether the region of convergence has $a \geq \text{ or } \leq \text{ vs. } > \text{ or } <$,
but realize that in some problem you work in the future it may matter, and you will have to be more careful.)

(a) 
\[ f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n} \]

(b) 
\[ \sum_{n=1}^{\infty} e^{nz} \]

(c) 
\[ \sum_{n=1}^{\infty} \frac{z^n}{n!} \]

2. (4 points) Use the integral test to prove the $p$-series test: “The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and
diverges if $p \leq 1.”$ Caution: Do the case for $p = 1$ separately.

3. (4 points) Find the Taylor series for
\[ f(z) = \frac{z^2}{z+2} \]

about a point $z_0 = 3$, and find the region of convergence for the Taylor series.

4. (6 points) Find the Maclaurin series for the following functions. You should notice that, by putting
the first two series together, you will get the third series, proving that Euler’s formula is true!

(a) $\cos(z)$

(b) $i \sin(z)$

(c) $e^{iz}$

5. (4 points) Now that we really trust Euler’s formula, find a way to write $\sin(z)$ and $\cos(z)$ in terms of
complex exponentials (i.e. with terms like $e^{iz}$ and $e^{-iz}$ and no trig functions).

6. (10 points) Let’s investigate sines and cosines of complex numbers.

(a) Start with your expression of $\sin(z)$ from the problem above. Now write $z$ in this expression as
$a + ib$.

(b) Separate the terms with $a + ib$ into terms which are products of things with just $a$ times things
with just $b$.

(c) Now use Euler’s formula to separate the expression you found above into a real and an imaginary
part.

(d) Now find an expression for the modulus (i.e., the absolute value) of $\sin(z)$. 
(e) Use Mathematica, MatLab, Maple or some other piece of software\(^1\) to plot the expressions you just found for the real and imaginary parts and the modulus of \(\sin(a + ib)\) from \(a = -2\pi\) to \(2\pi\) with \(b = 0\). Then do the same thing for \(b = \pi/4\), \(b = \pi/2\), and \(b = \pi\). Print your graphs and turn them in with your homework. You also might want to check your work by having the software compute \(\Re(\sin(z))\), \(\Im(\sin(z))\), and \(|\sin(x)|\) and plotting those as well.

7. (6 points) Consider, for example, the function

\[
f(z) = \frac{\sin(z)}{(z - i)^2}.
\]

At the location \(z = i\), this has a singularity. If we wrote this as a Taylor series about some point \(z_0\), the Taylor series would diverge beyond some radius because of the singularity. But a Laurent series expanded around the point \(z_0 = i\) will converge everywhere except right at the singularity. So a Laurent series can often be more useful than a Taylor series.

A Laurent series is very similar to a Taylor series, but in addition to having terms of the form \((z-z_0)^n\), it has terms of the form \((z-z_0)^{-n}\). Usually, when you learn about Laurent series, integrals are shown which can be used to calculate the coefficients in the series. The integrals are sometimes hard to calculate. But often you can do a simple transformation which makes it possible to generate a Laurent series using the methods of a Taylor series. Let’s do that with the above function.

(a) First, let’s let \(u \equiv z - i\). Write the function in terms of \(u\) rather than \(z\).

(b) Now you should have the sine of something over \(u\) to some power. Well, let’s ignore the \(u\) to some power part, and write a Taylor series of the sine part about the point \(u = 0\). Note that the sine doesn’t have a singularity, so this Taylor series is good everywhere.

(c) Now multiply your Taylor series by the part we left off - the one over \(u\) to some power part. Now you have a series that looks a lot like your Taylor series, but it now has negative powers of \(u\).

(d) Convert back to \(z\). Now you have the Laurent series for this equation - and it converges everywhere except right at the singularity! Yeah!

---

\(^1\)Note: All three of these packages should be installed on all of the computers in the department computer lab in N212. If you need an account for the computers in N212, just go to http://www.physics.byu.edu/ComputerSupport/ComputerAccounts.aspx.

If you would like to have math software on your own computer, I believe that there are educational versions of Mathematica, Maple, and MatLab which are not as expensive as the regular versions of the software (but I don’t think they are cheap). There’s also a free (but less flashy) program called GNU Octave. It has similar functionality to MatLab. Search for it on Google.