1. (15 points) In June of 2010 a lightning strike created a hole in an oil line, causing oil to leak into the Jordan river. Because the flow of the river causes the oil to move much faster than diffusion does, we can model the contamination of oil in the river with a PDE that has a “convection” term, but no diffusion term:

\[
\begin{align*}
\text{PDE} \quad u_t &= -vu_x, \quad 0 < x < \infty, 0 < t < \infty \\
\text{IC} \quad u(x,0) &= 0, \quad 0 \leq x < \infty \quad (\text{the water downstream is initially clean}) \\
\text{BC} \quad u(0,t) &= C, \quad 0 \leq t < \infty \quad (\text{the oil keeps the } x = 0 \text{ location contaminated})
\end{align*}
\]

where \( C \) and \( v \) are constants (\( v \) is the velocity of the water in the river).

(a) Find \( u(x,t) \) using a Laplace transform with respect to \( t \) to get an ODE problem, solving the ODE problem, and transforming back. Do not use a coordinate transform.

(b) Plot \( u(x,t) \) from \( x = 0 \) to 10 for \( t = 0, 3, 6, \) and 9. Let \( C = v = 1 \).

2. (15 points) James Bond is walking through a very very long subway tunnel (extending from \( x = -\infty \) to \( x = \infty \)) filled with fetid, but breathable, air. He notes that there is a breeze blowing through the tunnel. Using his super spy watch, he determines that the breeze is moving at a velocity \( v \) in the \(+x\) direction. Then, at time \( t = 0 \), Evil Dr. Durfee’s poison gas balloon pops, generating an initial gas distribution of

\[
\rho(x,0) = \rho_0 e^{-\sigma x^2}
\]

where \( \rho_0 \) and \( \sigma \) are deadly constants. James Bond knows that the distribution of poisonous gas will evolve according to the equation

\[
\rho_t = D\rho_{xx} - v\rho_x,
\]

where \( D \) is a constant. It is vital that Mr. Bond know what \( \rho(x,t) \) is from \( x = -\infty \) to \( \infty \) for all times \( t \geq 0 \). Otherwise he could end up dead, ruining an entire movie franchise.

(a) Transform the problem into a new coordinate system \( \spadesuit = x - vt \) and \( \clubsuit = t \). Never mind, that’s too hard to write. Let’s transform the problem into the new coordinate system \( \gamma = x - vt \) and \( \tau = t \). Since the initial condition is only valid at \( t = 0 \), it should be very simple to transform it into the new coordinate system. It’s only a little harder to transform the PDE - nothing a master spy can’t handle.

(b) Look at the PDE you just found above. Now look at the infinite diffusion problem we solved many days ago using Sine transforms. Now look back at the PDE you just found. Now look back at the problem we solved with Sine transforms. It’s not like the PDE you just found. But it could be if we just redefined a few constants. So let’s just copy that solution and fix it up to find \( \rho(\gamma,\tau) \). And let’s do it with style and finesse. In case you’ve forgotten, the problem was

\[
\begin{align*}
\text{PDE} : \quad \rho_{yy} &= \gamma \rho_t, \quad -\infty < y < \infty \\
\text{IC} : \quad \rho(y,0) &= \rho_0 e^{-\sigma y^2}, \quad -\infty < y < \infty
\end{align*}
\]

and the solution was

\[
\rho(y,t) = \frac{\rho_0}{2\sqrt{1/4 + t\sigma/\gamma}} e^{-y^2/(1/4 + t\gamma/\gamma)}.
\]

But I don’t have to tell you that, do I, Mr. Bond.
(c) Transform your solution back to find $\rho(x, t)$. Do this while heartlessly killing a bunch of guards who’s only crime was to be between you and the answer to this problem. At the end, make a witty comment to insinuate that you were not the least bit flustered by this problem or the moral dilemma of killing innocent people in pursuit of an important matter of mathematical security.

(d) Let $\sigma = D = \rho_0 = v = 1$ and plot $\rho(x, t)$ from $x = -15$ to $15$ at times $t = 0$, $2$, $4$, and $6$. Then exit the scene in a dramatic fashion, but with a calm demeanor and a bored expression on your face. Take the pretty girl with you.

3. (10 points) Consider sound waves on a windy day. The standard one-dimensional linear wave equation is

$$u_{xx} = \frac{1}{v^2} u_{tt},$$

where $v$ is the speed that waves travel. But if the medium that the wave is traveling in is moving, you get a different equation. That equation should transform into the standard 1-D linear wave equation if you transform into a frame moving with the medium. So we can figure out what the equation is for linear waves in 1-D with a moving medium by transforming the other way. If we assume that the medium is moving with a velocity $v_m$ in the $+x$ direction, we can start in a frame moving with the medium and then transform back into the "lab" frame by letting $\gamma = x + v_m t$ and $\tau = t$. Use this transformation on the wave equation above to find an equation for waves in a moving medium.