1. (15 points) Consider the following initial-value problem. Here $\gamma$, $T_0$, $R$, and $k$ are constants. Note that we would have had troubles using Fourier transforms or sine transforms to solve this one because the initial condition does not go to zero nicely as $x \to \infty$.

\[
\begin{align*}
PDE & \quad T_{xx} = \gamma T_t \quad 0 < x < \infty, 0 < t < \infty \\
BC & \quad T(0, t) = Rt \quad 0 < t < \infty \\
IC & \quad T(x, 0) = T_0 \sin kx \quad 0 < x < \infty
\end{align*}
\]

(a) Take that Laplace transform with respect to time of each side of the equation and the B.C. to convert this PDE problem into an ODE problem.

(b) Find the solution to this ODE, apply the transformed B.C., and note that $T(x, 0)$ is finite at all values of $x$ to find the constants in your solution.

(c) Now take the inverse transform of your solution to find $T(x, t)$. Note that for part of this you will need to use the Laplace transformation convolution theorem, and note that if $a$ and $b$ are positive real constants then

\[
\int_0^b \text{erfc} \left( \frac{a}{\sqrt{x}} \right) dx = -2a\sqrt{\frac{b}{\pi}} e^{-a^2/b} + (2a^2 + b)\text{erfc} \left( \frac{a}{\sqrt{b}} \right).
\]

(d) Let $\gamma = T_0 = k = R = 1$, and plot $T(x, t)$ from $x = 0$ to 30 for $t = 0$ and $t = 2$.

2. (10 points) So far we’ve only considered diffusion in one dimension. In two dimensions, the equation looks like

\[
T_{xx} + T_{yy} = \gamma T_t.
\]

Let’s imagine we are looking at one corner of an enormous square sheet of aluminum. We’ll let that corner be our origin with one edge running along the $x$ axis, and one running along the $y$ axis, such that the sheet extends from $x = 0$ to $x = \infty$ in the $x$ direction, and $y = 0$ to $y = \infty$ in the $y$ direction. Let’s assume that initially the sheet is at a temperature of zero everywhere, and that starting from time $t = 0$ we hold the $x = 0$ edge to a temperature $T_A$ and we hold the $y = 0$ edge at 0 temperature.

In other words

\[
T(0, y, t) = T_A
\]

and

\[
T(x, 0, t) = 0.
\]

(a) Perform the Laplace transform with respect to time on the PDE and the boundary conditions to find a PDE problem with just two variables.

(b) Use separation of variables (i.e. write your function as $f(x)g(y)$) to find the general form of the solution to the 2-variable PDE.

To finish this problem, we’d write the solution as a sum of functions of the form you found in (b), then find the unknown constants by fitting the function to the boundary conditions, then transform back. Or maybe we could do another transform to get rid of another variable. Maybe a sine transform this time? Anyway, I’m not going to make you do that today.
3. (15 points) Consider a semi-infinite insulated rod which is initially at a temperature of zero everywhere. Then at time $t = 0$ I start driving the temperature at one end such that $T(0, t) = T_A \sin(\omega t)$.

(a) Take the Laplace transform of the PDE ($T_{xx} = \gamma T_t$), and the boundary condition to get an ODE problem.

(b) Find the solution to the ODE, assume that the temperature stays finite as $x \to 0$, and apply your boundary conditions to find the solution to the transformed problem.

(c) Transform back to find $T(x, t)$. To do this, use the convolution theorem. And don’t bother actually evaluating the convolution integral - just write $T(x, t)$ in terms of the convolution integral.

(d) Let $\omega = 2\pi$, $T_A = 1$, and $\gamma = 1$, and numerically evaluate the integral and plot $T(x, t)$ from $x = 0$ to $x = 2$ at times $t = 0, 0.25, 0.5, 0.75,$ and $1$. 