1. Imagine that I look at an object in space through a telescope. Because the telescope has a finite aperture, things which should appear as tiny points instead appear as a blob in the form of the aperture’s diffraction pattern. For a circular opening, this pattern is known as the “Airy disk.” But, to keep things simple, let’s do this on one dimension - let’s assume that the aperture of our telescope is a slit. A slit has a diffraction pattern equal to

\[ I = I_0 \left( \frac{\sin(\beta)}{\beta} \right)^2, \]

where

\[ \beta = \frac{d \sin(\theta)}{\lambda}, \]

and \( d \) is the width of the slit, \( \theta \) describes the position in the image, and \( \lambda \) is the wavelength of the light. Of course, this assumes that the light source is located at \( \theta = 0 \). If, instead, the light source is located at some angle \( \gamma \), we just replace \( \theta \) with \( \theta - \gamma \) in the equation above.

This is illustrated in the plot below. In this plot, the blue line shows what we would get with an ideal telescope as we looked into the sky from an angle of \(-\pi/2\) radians to \(\pi/2\) radians from the position of the star. We would get no light intensity except right at the star’s position. The green line is what we get looking through our “single slit” telescope with \( d = 10\lambda \). Because of the diffraction of the light, the star is now somewhat blurry.

Knowing how the telescope behaves for a point source, we can find out what happens to the image of an extended object, because the extended object can be thought of as a sum of point sources. Each point object makes a pattern like the one up above, with \( I_0 \) proportional to \( d\gamma \) times the intensity in the “ideal” image at that particular point. So if I describe each point on the “ideal” image with a coordinate \( \gamma \), each point adds a contribution to the “blurred” image intensity at some location \( \theta \) equal to:

\[ dI = \Gamma I_{\text{ideal}}(\gamma) \left( \frac{\sin(d\sin(\theta - \gamma)/\lambda)}{d\sin(\theta - \gamma)/\lambda} \right)^2 d\gamma, \]

where \( \Gamma \) is the proportionality constant between \( I_0 \) and the intensity in the ideal image. (To find the proportionality constant, we need to compare how much light enters the telescope with the total integrated light in the image. For now, let’s just let \( \Gamma \) be an arbitrary constant.)
We can just integrate the equation above to find the total intensity pattern as a function of $\theta$.

(a) Imagine that we are looking at some object in the sky that, ideally, we would see as having an intensity profile of $I_{\text{ideal}}(\theta) = 1$ for values of $\theta$ from $-0.5$ radians to $0.5$ radians, and zero everywhere else. Make a plot of $I_{\text{ideal}}(\theta)$ from $\theta = -\pi/2$ to $\pi/2$.

(b) If we integrate $dI$ over all possible angles, we get $I$. This doesn’t look like something that you want to integrate, though, does it? So, let’s do it numerically in MatLab, Mathematica, etc. Numerically integrate and plot $I$ from $\theta = -\pi/2$ to $\pi/2$ assuming that $d/\lambda$ is 10, and letting $I$ be whatever constant real number you want. Attach a printout of your code and the plot. Note that the function to integrate numerically in MatLab is “quad,” and in Mathematica it is “NIntegrate.”

By the way, what you just did was a convolution integral. You convolved the ideal image, $I_{\text{ideal}}$ with the point-transfer-function of the telescope, $(\sin(\beta)/\beta)^2$ to get

$$f(x) \otimes g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi.$$ 

(Note that the convolution integral as defined above - and how it’s defined in your book - has a $1/\sqrt{2\pi}$ which in other sources might not be there. The definition we are using makes the convolution theorem, discussed below, work out nicer. But when we apply it to real, physical things, we have to multiply our results by $\sqrt{2\pi}$ to get rid of this $1/\sqrt{2\pi}$ and get physical results.)

When we worked this problem, we didn’t integrate from $-\infty$ to $\infty$, but that’s just because our functions weren’t defined over that whole range (we could just consider that the functions are zero outside of $\theta = -\pi/2$ to $\pi/2$).

2. Rather than working convolution integrals directly, you can also use Fourier transforms to evaluate convolutions. This is especially useful when I’m working with real, discrete, data, because I can use the speedy Fast Fourier Transform algorithm instead of more time consuming numerical integration. Let’s prove the convolution theorem, which relates convolution to Fourier transforms, by following the steps below.

(a) Write down

$$f(x) \otimes g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi.$$ 

(b) Now take the Fourier transform (the complex infinite one) of both sides of the equation (you should now have $\mathcal{F}[f(x) \otimes g(x)]$ on the left and a double integral on the right).

(c) For a double integral, we’re free to choose which integral to do first. Let’s plan on doing the $x$ integral first. So for now, think of $\xi$ as a constant. Then make the transformation $v = x - \xi$, and rewrite the double integral in terms of $v$ and $\xi$ (rather than $x$ and $\xi$).

(d) Now your exponential term should have a “plus sign” in it. Use this to split the exponential into two exponentials multiplied together. Then arrange the integrand so that the terms with $\xi$ in them are lumped together, and the terms with $v$ in them are lumped together.

(e) Note that the $\xi$ terms aren’t affected by $v$, and the $v$ terms aren’t affected by $\xi$. So we can bring one set of terms outside of one of the integral signs (because they are constant in that integral). Use this to write your double integral as the product of two single integrals.

(f) Now change $v$ and $\xi$ in each integral to $x$ ($v$ and $\xi$ are just parameters that get integrated away) and distribute the $1/2\pi$ term equally among the two integrals to make the right-hand-side look like the product of two Fourier transforms:

$$\mathcal{F}[f(x) \otimes g(x)] = \mathcal{F}[f(x)] \mathcal{F}[g(x)].$$
3. Let’s use the convolution theorem to see what will happen if I put a light signal with an intensity of \( \sin(\omega_{\text{light}}t) \) onto a photo detector with an impulse-response function of \( e^{-t^2}/\sqrt{\pi} \). The convolution integral for this problem is . . .

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\omega_{\text{light}}(t-\xi)) \frac{e^{-\xi^2}}{\sqrt{\pi}} d\xi
\]

(or we could have put the \( \xi \) into the sin term and the \( t - \xi \) into the exponential; it doesn’t matter which one you do). This integral is doable, but it is easier if we use the convolution theorem. So let’s do that instead.

(a) Find \( \mathcal{F}[\sin(\omega_{\text{light}}t)] \). We can do this without actually doing any integrals . . . Since this is a pure sine wave, we know that if I did a cosine/sine Fourier transform, we would get \( a(\omega) = 0 \). And we know that \( b(\omega) \) has to be zero everywhere except at \( \omega = \omega_{\text{light}} \). We also know that the inverse transform should give us the sine back:

\[
\int_0^{\infty} b(\omega) \sin(\omega t) d\omega = \sin(\omega_{\text{light}}t).
\]

If you look at the equation above, remember that \( b(\omega) \) is zero except at \( \omega_{\text{light}} \), and if you are familiar with the Dirac delta function, the answer should be staring you in the face. If you aren’t familiar with the Dirac delta function, look at the footnote below3. Once you figure out what \( b(\omega) \) should be, you should be able to convert from \( a(\omega) \) and \( b(\omega) \) to \( c(\omega) \).

(b) Find \( \mathcal{F}[e^{-t^2}] \). Note that

\[
\int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt = \sqrt{\pi} e^{-\omega^2/4}.
\]

(c) Multiply the two of them together and do the inverse transform to get the actual response of the photo detector to the light. Using the properties of the Dirac delta function, the inverse transform should be trivial. You should be able to write this such that it is obvious that it is real (with no \( i \)’s in it).

(d) Multiply your answer by \( \sqrt{2\pi} \) to get a physical, meaningful result - what will actually come out of the photo detector (see the comment in problem 1 about the issue with \( \sqrt{2\pi} \)). In the limit of small \( \omega \), the finite response time of our photo detector should not be an issue. Do you get \( \sin(\omega_{\text{light}}t) \) back in the limit as \( \omega \to 0 \)?

4. Derive the following identities:

(a) \( \mathcal{F}[u_x] = i\omega \mathcal{F}[u] \).

(b) \( \mathcal{F}[u_t] = \frac{\partial \mathcal{F}[u]}{\partial t} \).

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3The Dirac delta function, \( \delta(x) \), is equal to zero everywhere except at \( x = 0 \), where it is equal to infinity. It is defined such that if you integrate \( \int \delta(x) dx \) across \( x = 0 \), you will get 1. Also note that if you integrate \( \int f(x) \delta(x) dx \), the integrand is zero everywhere except at \( x = 0 \). At \( x = 0 \), any reasonable function \( f(x) \) will change slowly over the infinitesimal range where \( \delta(x) > 0 \). So over that range, it acts like a constant, and \( \int f(x) \delta(x) dx = f(0) \int \delta(x) dx \).