

# 1 The Diffraction Grating

Okay, so Young used the interference from a pair of slits to prove that light is a wave. What else can you do with a double-slit interference pattern? For one thing, you can measure the wavelength of light. But since the lines in the interference pattern aren't infinitely thin, there is a limit to how well you can determine the light's wavelength. Applying Rayleigh's criterion, if I put two colors of light onto my grating, I'll be able to tell pretty easily that they are two separate colors if the interference maximum of one wavelength fall on a minimum of the other wavelength.

## 1.1 Minimums and maximums in the two-slit interference pattern

The locations of bright fringes are given by the relation

$$\sin(\theta_{max}) = m \frac{\lambda}{d},$$

and the locations of dark fringes are given by

$$\sin(\theta_{min}) = (m + \frac{1}{2}) \frac{\lambda}{d}.$$

I can just resolve the difference between a wavelength  $\lambda_a$  and a shorter wavelength  $\lambda_b$  if  $\lambda_a$  has maximum at the same angle that  $\lambda_b$  has a minimum:

$$\theta_{max a} = \theta_{min b} \Rightarrow \sin(\theta_{max a}) = \sin(\theta_{min b}).$$

This implies that

$$m \frac{\lambda_a}{d} = \left(m + \frac{1}{2}\right) \frac{\lambda_b}{d} \Rightarrow \frac{m}{d} (\lambda_a - \lambda_b) = \frac{1}{2} \frac{\lambda_b}{d}.$$

This means that the smallest difference in wavelengths I can measure is

$$\Delta\lambda = \frac{\lambda_b}{2m}.$$

## 1.2 The $N$ -slit interference pattern

If we increase the number of slits, our ability to make precise measurements of wavelength and resolve similar wavelengths of light improves. In the limit as  $N$  becomes large, we call our mask with many slits a "diffraction grating."

### 1.2.1 Resolving power of a grating: hand waving approach

For a two slit pattern, the distance from a maximum to the nearest minimum is half the distance from the maximum to the next maximum. But in an  $N$  slit pattern the distance is smaller. As we found earlier, an  $N$ -slit pattern has bright fringes at the same location as a two-slit pattern. But in between each large maximum are  $N - 2$  smaller, sub maxima. So instead of dipping to zero once between large maxima, the intensity drops  $N - 1$  times. If we assume that the sub-maxima fringes are just as wide as the large maxima fringes, the distance to the nearest minimum in an  $N$ -slit pattern is  $1/(2N - 2)$  the distance to the next maximum. This would imply that in an  $N$ -slit pattern, the distance from a large maximum to the next minimum should be about  $1/(N - 1)$  times the distance for a two-slit pattern. The smallest resolvable  $\Delta\lambda$  should be reduced by the same factor.

This gives you an intuitive idea as to why the resolving power increases as the number of slits increases. But while this approach is qualitatively correct, it is not *quantitatively* correct. We made an assumption in this derivation - that all of the sub-maxima fringes were the same width as the large maxima fringes - which isn't quite valid. To get the right answer, we need to do a more rigorous calculation.

### 1.2.2 Resolving Power of a grating: rigorous treatment

In an  $N$ -slit pattern you get the brightest fringe when all of the slits are in phase. This happens when the phase difference between contributions from adjacent slits is an integer times  $2\pi$ :

$$\Delta\phi_{large max} = 2\pi m.$$

You get a minimum when we increase  $\Delta\phi$  just enough to close the loop, as illustrated in Fig. 1. In other words, if we start with  $\Delta\phi_{large max}$  and add some additional phase shift  $\delta\phi$  between adjacent slits, we will get the first minimum after

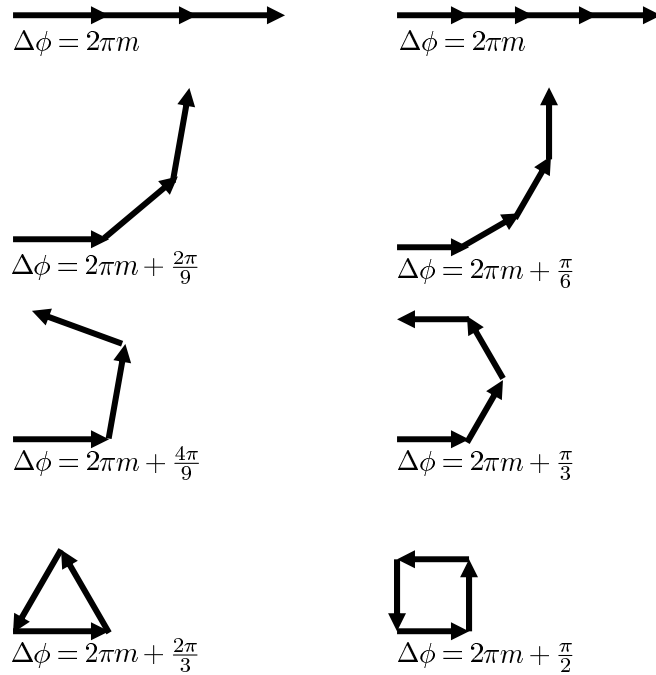


Figure 1: Phasor diagrams for a three- and four-slit interference pattern. The phasor diagram for a three-slit pattern is shown on the left, and the diagram for a four-slit pattern is shown on the right. At the top, all of the phasors add in phase to produce a maximum. Moving down, the phase between contributions from adjacent slits,  $\Delta\phi$ , increases. In the bottom frames,  $\Delta\phi$  is just large enough to make the first dark fringe past the  $m^{\text{th}}$  large bright fringe.

the large maximum when  $N\delta\phi$  is equal to  $2\pi$ . So the phase shift past the large maximum  $\delta\phi$  which produces the first minimum is

$$\frac{2\pi}{N}.$$

Since  $\Delta\phi = 2\pi d \sin(\theta)/\lambda$ , the angle at which we get the big maximums is given by the equation

$$\sin(\theta_{\text{large max}}) = m \frac{\lambda}{d},$$

and the equation for the first minimum past the  $m^{\text{th}}$  maximum is

$$\sin(\theta_{\text{1st min}}) = m \frac{\lambda}{d} + \frac{\lambda}{Nd}$$

So I can just barely resolve the difference between a wavelength  $\lambda_a$  and a shorter wavelength  $\lambda_b$  if  $\lambda_a$  has maximum at the same angle that  $\lambda_b$  has a minimum:

$$\theta_{\text{large max } a} = \theta_{\text{1st min } b} \Rightarrow \sin(\theta_{\text{large max } a}) = \sin(\theta_{\text{1st min } b}).$$

So this means that

$$m \frac{\lambda_a}{d} = m \frac{\lambda_b}{d} + \frac{\lambda_b}{Nd} \Rightarrow m(\lambda_a - \lambda_b) = \frac{\lambda_b}{N}.$$

So

$$\Delta\lambda = \frac{\lambda_b}{mN}.$$

Of course, in the limit of large  $N$ , we get a small  $\Delta\lambda$ , so  $\lambda_b \approx \lambda_a$ . As such instead of writing  $\lambda_b$ , I'll just write  $\lambda$ . This makes our equation

$$\Delta\lambda \approx \frac{\lambda}{mN},$$

and the resolving power is

$$R \equiv \frac{\lambda}{\Delta\lambda} = mN.$$