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Physics 123 Term Project

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12 December 2001

Oscillator Motion

Introduction

The purpose of this project is to determine the motion of an oscillator with known damping, spring constant, initial position, and initial speed. Assume a massless spring with spring constant k with a mass m attached to one end, while the other end is fixed. The spring is damped in a medium with dampening coefficient $\gamma 1$ exerting a force $\gamma 1 \left(\frac{\partial}{\partial t} \mathrm{u}(t)\right)$; this investigation will examine oscillators with no external forces acting on them.

Derivation of oscillator's position equation

```
> restart;
```

Given a spring with a position $\mathrm{u}(t)$ at time t and with initial position $\mathrm{u}(0) = u_0$ and initial velocity $\frac{\partial}{\partial t} \mathrm{u}(0) = v_0$. We will use Newton's second law of motion, $\sum \mathrm{F}(t) = m \left(\frac{\partial^2}{\partial t^2} \mathrm{u}(t)\right)$, to derive an expression for the position of the oscillator.

The first force on the spring is caused by gravitational force on the attached mass, given by:

```
> Fg:=m*g;
```

$$Fg := m\, g$$

The second force on the spring is the spring force, given by Hooke's law:

```
> Fs:=-k*(L+u(t));
```

$$Fs := -k\, (L + \mathrm{u}(t))$$

where L is the displacement of the spring from its position without a connected mass to its equilibrium position with the mass.

The third force acting on the spring-mass system is a damping or resistive force F_d, caused by air, water, or whatever other medium in which the oscillation occurs:

```
> Fd(t):=-gamma1*diff(u(t),t);
```

$$\mathrm{Fd}(t) := -\gamma 1 \left(\frac{\partial}{\partial t} \mathrm{u}(t)\right)$$

The fourth and final force affecting this system is an external force $\mathrm{F}(t)$--this force will be assumed to be zero throughout this project:

```
> F(t):='F(t)';
```

$$\mathrm{F}(t) := \mathrm{F}(t)$$

Substituting into Newton's law ($\Sigma F = m\,a$) gives:

```
> eq:=Fg+Fs+Fd(t)+F(t)=m*diff(u(t),t$2);
```

$$eq := m\,g - k\,(L + \mathrm{u}(t)) - \gamma 1 \left(\frac{\partial}{\partial t}\mathrm{u}(t)\right) + \mathrm{F}(t) = m\left(\frac{\partial^2}{\partial t^2}\mathrm{u}(t)\right)$$

This can be rewritten in the following form when solve for $\mathrm{F}(t)$:

```
> eq:=F(t)=solve(eq, F(t));
```

$$eq := \mathrm{F}(t) = -m\,g + k\,L + k\,\mathrm{u}(t) + \gamma 1\left(\frac{\partial}{\partial t}\mathrm{u}(t)\right) + m\left(\frac{\partial^2}{\partial t^2}\mathrm{u}(t)\right)$$

We defined L in such a way that $m\,g - k\,L = 0$, so we will eliminate that expression from the equation:

```
> eq:=F(t)=k*u(t)+gamma1*diff(u(t),t)+m*diff(u(t),`$`(t,2));
```

$$eq := \mathrm{F}(t) = k\,\mathrm{u}(t) + \gamma 1\left(\frac{\partial}{\partial t}\mathrm{u}(t)\right) + m\left(\frac{\partial^2}{\partial t^2}\mathrm{u}(t)\right)$$

We'll also assume that m, $\gamma 1$, and k are all positive.

We will now solve this second-order differential equation using Maple, assuming the initial position u_0 and velocity v_0:

```
> F(t):=0; sol:=dsolve({eq, u(0)=u0, D(u)(0)=v0},u(t));
```

$$\mathrm{F}(t) := 0$$

$$sol := \mathrm{u}(t) = \frac{1}{2}\,\frac{e^{\left(-1/2\,\frac{(\gamma 1 - \sqrt{\gamma 1^2 - 4\,k\,m})\,t}{m}\right)}\,(\gamma 1\,u0 + \sqrt{\gamma 1^2 - 4\,k\,m}\,u0 + 2\,v0\,m)}{\sqrt{\gamma 1^2 - 4\,k\,m}} - \frac{1}{2}\,\frac{e^{\left(-1/2\,\frac{(\gamma 1 + \sqrt{\gamma 1^2 - 4\,k\,m})\,t}{m}\right)}\,(\gamma 1\,u0 - \sqrt{\gamma 1^2 - 4\,k\,m}\,u0 + 2\,v0\,m)}{\sqrt{\gamma 1^2 - 4\,k\,m}}$$

Specific solutions of oscillator equation

No dampening

After assigning values for m, γ, k, and the integration constants, we can graph this position function to see how the oscillator behaves. We'll initially assume that no external force acts on the spring.

We will now look at the path of an oscillator with no dampening (i.e. $\gamma = 0$). Will will assume no initial velocity and some initial displacement away from equilibrium. The oscillator's path should resemble a cosine wave with constant amplitude.

```
> m:=1/2; gamma1:=0; k:=60; F(t):=0;
```

$$m := \frac{1}{2}$$

$$\gamma 1 := 0$$

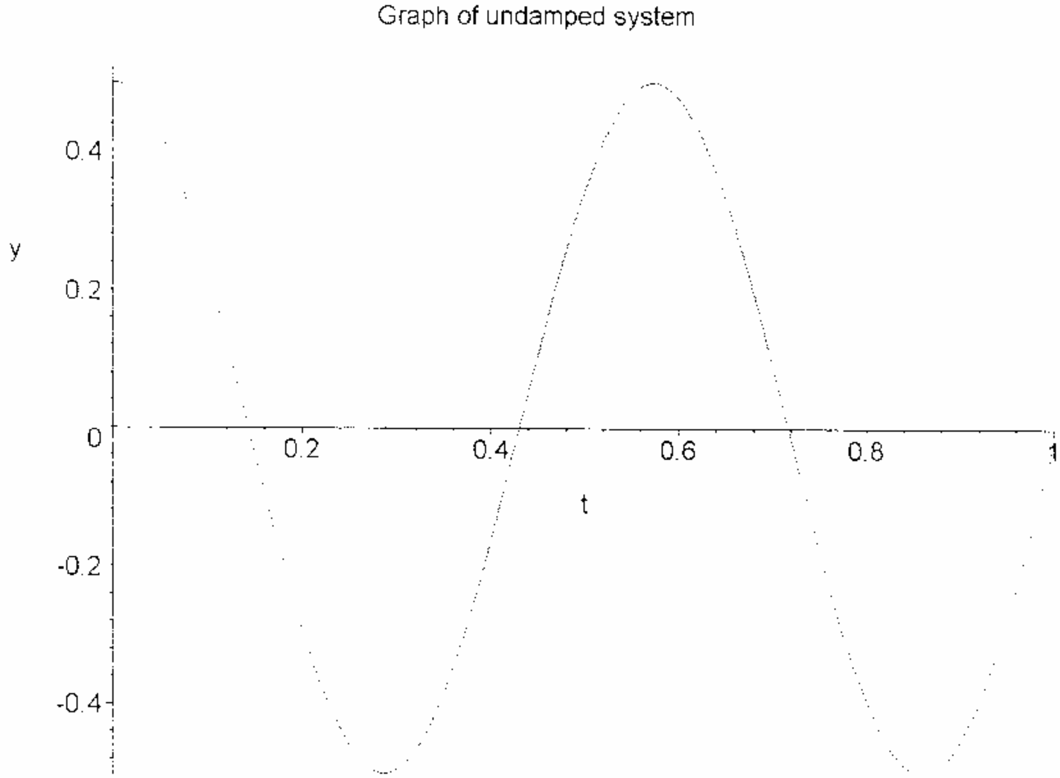
$$k := 60$$

$$F(t) := 0$$

```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=0},u(t)));
```

$$sol := \frac{1}{2}\cos(2\sqrt{30}\,t)$$

```
> plot(sol,t=0..1,y,title="Graph of undamped system");
```



Observe that the oscillator will continue this path indefinitely in the absense of dampening forces. Changing the initial velocity will shift the phase of the position function.

Dampening

When $\gamma \neq 0$, then the system is damped. Oscillations will die off at a rate that depends on the type of oscillator and what it is being oscillated in (the medium's dampening force is expressed by γ). There are three types of dampening: underdampening, overdampening, and critical dampening. A critically-damped system is on the threshold of the other two damped systems, which are at $\gamma = 2\sqrt{k\,m}$. We will now predict what values of m, γ, k and F(t) cause which case of dampening.
The type of dampning will be determined by the characteristic equation derived from the homogeneous differential equation that describes the motion of the oscillator, which is: $m\,r^2 + \gamma\,r + k = 0$. Solving for r, the equation becomes $r = \frac{-\gamma + \sqrt{\gamma^2 - 4\,m\,k}}{2\,m}$ or $r = \frac{-\gamma - \sqrt{\gamma^2 - 4\,m\,k}}{2\,m}$. So, the expression that determines what type of solutions the wave equation will have is $\gamma^2 - 4\,m\,k$. When this expression is zero, a root of multiplicity two is found. and the solution will be in the form $y = c_1\,e^{(r\,t)} + c_2\,t\,e^{(r\,t)}$, a

critically damped solution with no oscillations. When the expression is positive, the solution takes the form $y = c_1 e^{(r_1 t)} + c_2 e^{(r_2 t)}$, an overdamped solution with no oscillations. When the expression is negative, complex roots of r result, and the solution takes the form

$y = c_1 e^{(a_1 t)} \cos(b_1 t) + c_2 e^{(a_2 t)} \sin(b_2 t)$, where a and b are the real and imaginary parts of r, respectively. This oscillatory solution is characteristic of an underdamped system.

Let's take a look at each one of these situations:

(1) Underdamped

We'll use a large spring constant to overcome dampening forces in this underdamped system.

```
> m:=1/8; gamma1:=1; k:=24; F(t):=0;
  determinant:=(a,b,c)->b^2-4*a*c;
  Determinant=determinant(m,gamma1,k);
```

$$m := \frac{1}{8}$$

$$\gamma 1 := 1$$

$$k := 24$$

$$F(t) := 0$$

$$determinant := (a, b, c) \to b^2 - 4\, a\, c$$

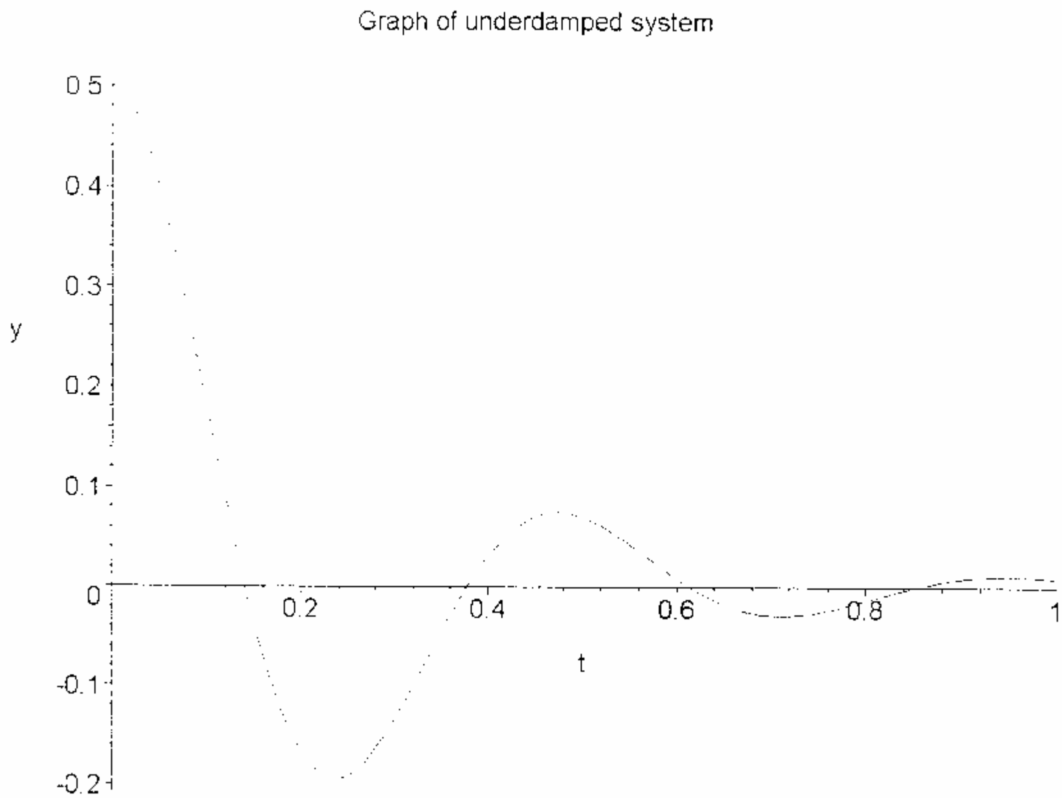
$$Determinant = -11$$

We can see from the determinant that the solution should be underdamped. A plot of the solution will verify this.

```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=0},u(t)));
```

$$sol := \frac{1}{22}\sqrt{11}\, e^{(-4\, t)} \sin(4\sqrt{11}\, t) + \frac{1}{2} e^{(-4\, t)} \cos(4\sqrt{11}\, t)$$

```
> plot(sol,t=0..1,y,title="Graph of underdamped system");
```



Observe how this underdamped oscillator passes the equilibrium position of the spring several times before the oscillations fade away.

(2) Critically damped

Only certain combinations of masses and spring constants result in critically damped systems. We'll give the oscillator no initial velocity and then a negative intial velocity to demonstrate two possible scenarios for its motion.

```
> m:=1; gamma1:=1; k:=0.25; F(t):=0;
  Determinant=determinant(m,gamma1,k);
```

$$m := 1$$

$$\gamma 1 := 1$$

$$k := .25$$

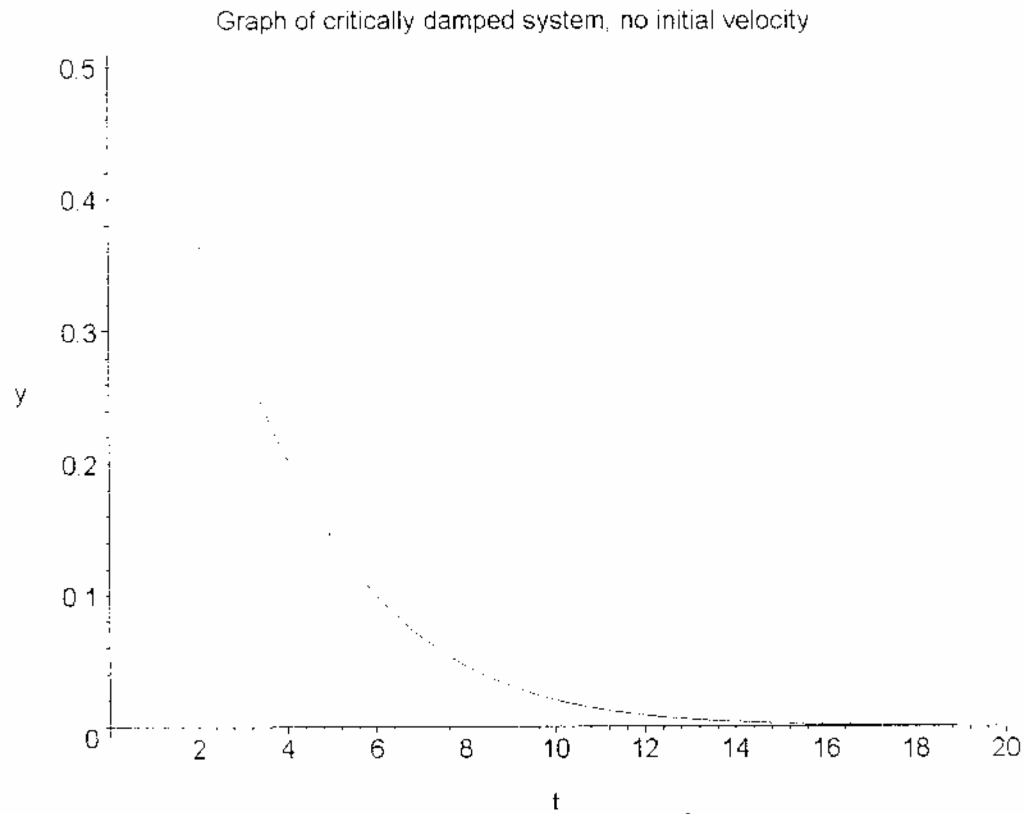
$$\mathrm{F}(t) := 0$$

$$Determinant = 0.$$

```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=0},u(t)));
```

$$sol := \frac{1}{2}\,\mathrm{e}^{(-1/2\,t)} + \frac{1}{4}\,\mathrm{e}^{(-1/2\,t)}\,t$$

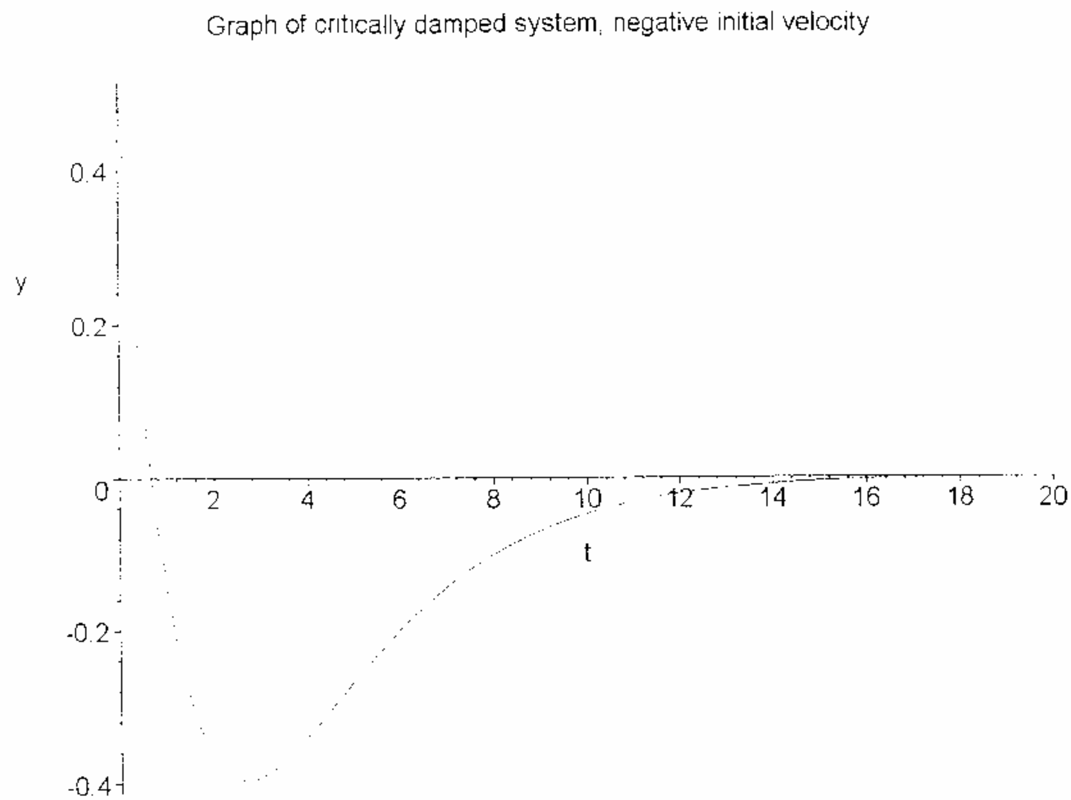
```
> plot(sol,t=0..20,y,title="Graph of critically damped system, no
  initial velocity");
```




```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=-1},u(t)));
```

$$sol := \frac{1}{2}e^{(-1/2\,t)} - \frac{3}{4}e^{(-1/2\,t)}\,t$$

```
> plot(sol,t=0..20,y,title="Graph of critically damped system,
  negative initial velocity");
```



Study carefully each graph and compare them to the overdamped solutions below. Notice that the overdamped solutions return to the equilibrium position relatively quickly.

(3) Overdamped

A large dampening constant will keep a mass from oscillating about the equilibrium point.

```
> m:=10; gamma1:=7; k:=1; F(t):=0;
  Determinant=determinant(m,gamma1,k);
```

$$m := 10$$

$$\gamma 1 := 7$$

$$k := 1$$

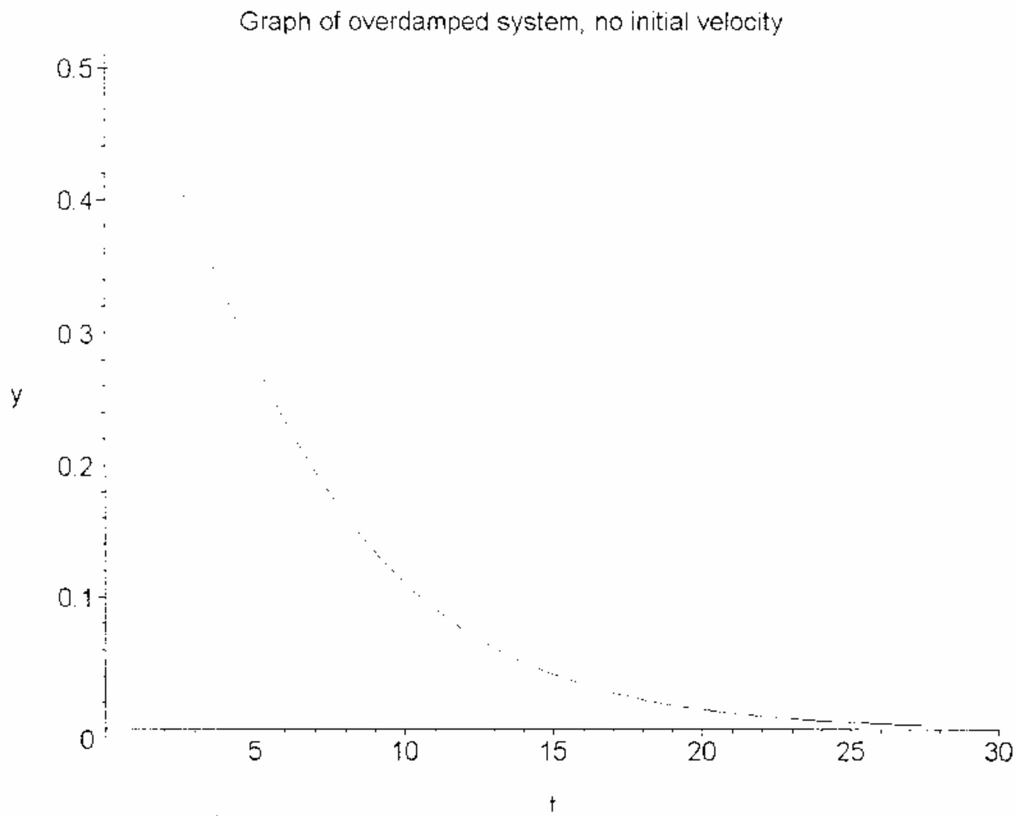
$$\mathrm{F}(t) := 0$$

$$Determinant = 9$$

```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=0},u(t)));
```

$$sol := \frac{5}{6}\mathrm{e}^{(-1/5\,t)} - \frac{1}{3}\mathrm{e}^{(-1/2\,t)}$$

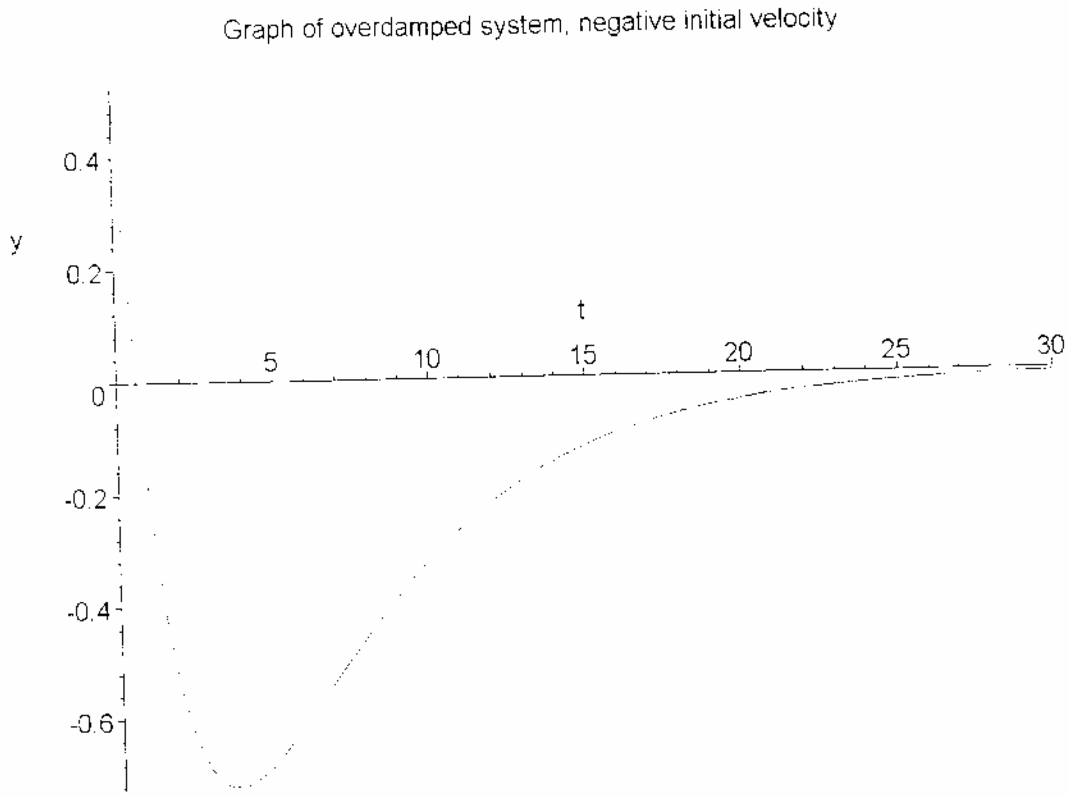
```
> plot(sol,t=0..30,y,title="Graph of overdamped system, no initial
  velocity");
```




```
> sol:=rhs(dsolve({eq, u(0)=1/2, D(u)(0)=-1},u(t)));
```

$$sol := -\frac{5}{2}\,\mathbf{e}^{(-1/5\,t)} + 3\,\mathbf{e}^{(-1/2\,t)}$$

```
> plot(sol,t=0..30,y,title="Graph of overdamped system, negative
  initial velocity");
```



Theses solutions are somewhat similar to the critically damped solutions, particularly in the way that the oscillator never crosses the equilibrium position more than once.

Conclusion

The spring constant, attached mass, and dampening coefficient affect whether an oscillator's motion follows a nondamped, underdamped, critically damped, or overdamped characteristic path. The expression $\gamma^2 - 4\, m\, k$ can be used to determine which of these paths the oscillator will take, where γ is the dampening coefficient, m is the mass of the object attatched to the spring, and k is the spring constant. The initial position and velocity of the oscillator will shift, stretch, or invert the curve, but not change its characteristic shape based on dampening. Further investigation could determine how external forces affect the oscillator, for example, how beats are formed and what conditions create resonance.