\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x) \]

If \( V \) is periodic: \( V(x+na) = V(x) \)

\[ \psi_{x+a} = \frac{1}{a} \psi_x \]

Then: (perhaps) \( \psi(x+na) = \psi(x) \) ?

No, so too strong a condition

\[ \psi(x+na) = e^{i k(a - x)} \psi_x \]

off by a phase factor

is OK, since \( |\psi|^2 \) is well-defined!

\[ \text{Actual Thin: } \quad \psi(x) = e^{ikx} \phi_k(x) \quad \text{with } \phi_k(x) = U_k(x+na) \]

Demonstrate: \( \psi(x+na) = e^{i k (n a)} \phi_k(x) = e^{i k (n a)} \phi_k(x+na) \)

\[ = \psi(x) \cdot e^{i k n a} \checkmark \text{ it worked!} \]

(That's half the problem anyway.

Still need to prove other relations; only this formula.)
Given Bloch theorem \( \Phi = \mathbf{u}_k(x) e^{i(k \cdot a)} \)

- Periodic \( \mathbf{u}_k(x) = \mathbf{u}(x + \mathbf{a}) \)
- Different for each wavenumber

Every wavenumber is associated with some free electron wavefunction \( \Psi \)

Can label every wavenumber \( k \) with \( n \) to obtain 1 electron

Consider wavefunction labeled \( n / k' = c \cdot k \)

\( \Psi_0 \) is in \( 1 \text{st} BZ \)

\[ \Psi_0(x) = e^{i(\frac{2\pi n x}{a})} \]

\( n \) is the label for \( k \)

Block function with \( k' \) inside \( 1 \text{st} BZ \)

Different \( n \) function thus label \( n \) to distinguish \( \Psi_0(x) \) from \( \mathbf{u}_k(x) \)

Since all electrons can be labeled with wave vector \( k \) inside \( 1 \text{st} BZ \), we can restrict \( x \) to \( 1 \text{st} BZ \) in all plots.

(\( n \) becomes index of band)
Also, the phase, \( \phi \), is separated by \( \frac{2\pi}{L} \)

\( \leftarrow \text{length of physical crystal} \)

\[ A \sin \left( \frac{2 \pi x}{L} \right) \rightarrow k = \frac{2\pi n}{L} \]

**Proof 1:**

For \( f = 0 \) in body

Then its "periodic in whole space well"

\( \text{(even number well)} \)

\[ y \]

Spacing is \( \frac{2\pi}{L} \)

But this includes possible values of \( k \)

To include maximum \( k \) or \( k + \frac{\pi}{L} N \)

must "switch" not by factor of 2

**Proof 2**

Force periodic body considering (because surface is unimportant)

\[ f(x+L) = f(x) \]

\[ y(x) = Ae^{2\pi i L} \rightarrow k = \frac{2\pi n}{L} \]

Spacing is \( \frac{2\pi}{L} \checkmark \)

Includes both + and - values of \( k \)

and direction of \( k \) can be used to indicate electric velocity.