Electrodynamics Exam 3 and Final Exam – Sample Exam Problems
Dr. Colton, Spring 2016

Note—In a few of the problems below, you need to know the following integral:

\[
\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2 \sqrt{a^2 + x^2}}
\]

Problems

1. Find the magnetic field at the “×” if a current, \( I \), flows in the directions indicated by the arrows (the × is right in the middle of the “2l” segment)

2. A loop of wire is formed into the shape of a rectangle with sides of length \( a \) and \( b \) \((a > b)\). Find the magnetic field (both magnitude and direction) at the center of the rectangle if the wire carries a current \( I \) that flows in the counterclockwise direction.

3. A current flows counterclockwise in a square loop of side length 2\( a \) that lies flat in the plane of this page. Determine the magnetic field at a point a distance \( L \) to the right of the center of the loop where \( L > a \).

4. A loop of wire is formed into the shape of a circle of radius \( R \) lying in the \( x-y \) plane, and carries a current \( I \) that flows in the counterclockwise direction when viewed from above. Find the magnetic field (both magnitude and direction) as a function of \( x \) and \( y \) for an arbitrary point in the \( x-y \) plane, for distances far away from the loop, \( \sqrt{x^2 + y^2} >> R \).
6. An infinitely long, thick cylindrical shell of inner radius \( a \) and outer radius \( b \) carries a current given by the combined effects of a volume current density, \( J = k_1 s^2 \hat{z} \) and a surface current density located at \( s=b \): \( K = k_2 b \hat{z} \). (There is no current inside \( s=a \).) Find \( B \) everywhere due to these current densities.

![Diagram of a cylindrical shell with J and K arrows](image)

7. A circular torus of radius \( a \) is wound tightly with \( N \) loops of wire each carrying a current \( I \). Find the magnetic field at the center of the torus’s cross section (the point marked “\( \times \”)”.

![Diagram of a torus with current loop](image)

8. An infinitely long cylinder of radius \( R \) has a magnetization of \( M = k/R \hat{z} \). Find the magnetic field everywhere due to this cylinder (a) via bound currents and Ampere’s law for \( B \), and (b) via Ampere’s law for \( H \).

![Diagram of a cylinder with M arrow](image)
9. An infinitely long, thick cylindrical shell of inner radius $a$ and outer radius $b$ has a magnetization of $M = k/s^2 \hat{\phi}$. Calculate the magnetic field as a function of distance from the center of the cylinder.

10. An infinitely long thick cylindrical shell of inner radius $a$ and outer radius $b$ carries a magnetization of $M = ks^2 \hat{z}$. Inside the cylinder is empty space.
   (a) Find the magnetic field everywhere using the bound currents.
   (b) Find the magnetic field everywhere using the $H$-field.

11. An infinitely long cylinder of radius $R$ has a magnetization of $M = k/s \hat{z}$. Find the magnetic field everywhere due to this cylinder.

12. A metal cylinder with a magnetic susceptibility of $\chi_m$ is bent into a circular shape and the ends are welded together to form a closed loop in the shape of a torus (a torus is donut-shaped object). Wire is wrapped tightly around the surface of the torus so that it is evenly covered by the $N$ windings. If the wire carries a current $I$, calculate the magnetic field inside and outside the torus in terms of given quantities and fundamental constants.

13. An infinitely long cylinder of radius $R$ centered on the $z$-axis is magnetized by a free current such that its magnetization is $M = k/s \hat{\phi}$. The magnetization is such that $B = 0$ everywhere. Find the free current. What is $\chi_m$? What type of magnetic material is this cylinder made of?

14. A solid infinite cylinder of radius $R$ is placed so that its axis coincides with the $z$-axis. A free current density is flowing in the cylinder, $J_f = k s^2 \hat{z}$. This current produces a magnetization in the cylinder of $\alpha s^3 \hat{\phi}$. Assuming the cylinder is made of a linear magnetic material:
   (a) Find the magnetic susceptibility of the cylinder in terms of given quantities and fundamental constants.
   (b) Find the magnetic field inside the cylinder.

15. An insulated wire is wrapped around a long thin metal cylinder having magnetic susceptibility $\chi_m$. It’s wrapped as a solenoid having $n$ turns per length, and a current $I$ is sent through the wire. Use what your book calls the “boundary conditions,” which is what I’ve been calling “the discontinuity in B” (and/or H), to determine the strength of the magnetic field in the air right outside the end of the cylinder. You may assume that the field inside the solenoid is the same as the case for an infinite solenoid.

16. A circular loop (radius $R$) is centered at the origin, and originally positioned such that it lies in the $x$-$z$ plane. There is a constant magnetic field present in the area, $B = B \hat{x}$. The loop is then rotated about the $z$-axis at a frequency $f$. Find the induced voltage in the loop as a function of time.

**Note:** the next three problems all involve the displacement current, and are hence not applicable for exam 3 but could be representative of final exam problems from Chapter 7.
17. A parallel plate capacitor of two circular plates of area $A$ has vacuum between the plates. This capacitor is connected to a battery of constant voltage, $V_0$. The plates are then slowly oscillated so that the distance between them is given by $d = d_0 + d_1 \sin \omega t$. Find $H$ between the plates due to the displacement current.

18. A parallel plate capacitor of two circular plates of area $A$ has a dielectric with permittivity $\varepsilon$ between the plates. This capacitor is connected to a battery of constant voltage $V_0$. The plates are slowly oscillating so that the distance between them is given by $d = d_0 + d_1 \sin \omega t$. Find $H$ between the plates due to the displacement current.

19. The following fields exist in a region of vacuum where $0 < x < a$, $\omega t$,

$$E_x = -\frac{\mu_0 \alpha a H_0}{\pi} \sin \left( \frac{\pi x}{a} \right) \sin(kz - \omega t)$$

$$H_y = \frac{ka H_0}{\pi} \sin \left( \frac{\pi x}{a} \right) \sin(kz - \omega t)$$

$$H_z = H_0 \cos \left( \frac{\pi x}{a} \right) \cos(kz - \omega t)$$

All other components are zero. Find the free charge density, free current density, and the displacement current in this region. $H_0$, $a$, $k$, and $\omega$ are constants and $\mu_0 \varepsilon_0 \omega^2 = k^2 + \pi^2 / a^2$. (Hint: Use the differential form of Maxwell's Equations.)
Answers (not 100% guaranteed; let me know if you detect any errors)

1. $\vec{B} = \frac{\mu_0 I}{\sqrt{2\pi d}}$ into the page

2. $\vec{B} = \frac{2\mu_0 I}{\pi} \frac{1}{\sqrt{a^2 + b^2}} \left( \frac{a}{b} + \frac{b}{a} \right)$ out of the page

3. $\vec{B} = \frac{\mu_0 I}{2\pi} \left[ \frac{1}{\sqrt{(L+a)^2 + a^2}} \left( \frac{a}{L+a} + \frac{L+a}{a} \right) \frac{1}{\sqrt{(L-a)^2 + a^2}} \left( \frac{a}{L-a} + \frac{L-a}{a} \right) \right]$ out of the page.

4. $\vec{B} = \frac{\mu_0 I R^2}{4(x^2 + y^2)^{3/2}} \hat{z}$

5. $\vec{B} = 0$ for $s<a$; $\vec{B} = \frac{\mu_0 k \ln(s/a)}{s} \hat{\phi}$ for $a<s<b$; $\vec{B} = \frac{\mu_0 k \ln(b/a)}{s} \hat{\phi}$ for $s>b$

6. $\vec{B} = 0$ for $s<a$; $\vec{B} = \frac{\mu_0 k}{4s} (s^4 - a^4) \hat{\phi}$ for $a<s<b$; $\vec{B} = \frac{H_0}{s} \left[ \frac{k_1}{4} (b^4 - a^4) + k_2 b^2 \right] \hat{\phi}$ for $s>b$

7. $\vec{B} = \frac{\mu_0 NI}{2\pi a} \hat{\phi}$

8. $\vec{B} = \frac{\mu_0 k}{s} \hat{z}$ for $s<R$; $\vec{B} = 0$ for $s>R$ (the same answers using either method)

9. $\vec{B} = 0$ for $s<a$; $\vec{B} = \frac{\mu_0 k}{s^2} \hat{\phi}$ for $a<s<b$; $\vec{B} = 0$ for $s>b$

10. $\vec{B} = 0$ for $s<a$; $\vec{B} = \mu_0 k s^2 \hat{z}$ for $a<s<b$; $\vec{B} = 0$ for $s>b$ (the same answers using either method)

11. $\vec{B} = \frac{\mu_0 k}{s} \hat{z}$ for $s<R$; $\vec{B} = 0$ for $s>R$

12. $\vec{B} = \frac{\mu_0 (1 + \chi_m) NI}{2\pi s} \hat{\phi}$ inside, $B = 0$ outside

13. $I = 2\pi k$ located at the origin (middle of the cylinder), in the $-\hat{z}$ direction; $\chi_m = -1$; it’s diamagnetic (but actually, I don’t think diamagnetic materials with such a large $\chi_m$ exist).
14. $\chi_m = \frac{4\alpha}{k} \; ; \; \vec{B} = \mu_0 s \left( k + \alpha \right) \hat{\phi}$, inside

15. $\vec{B} = \mu_0 (1 + \chi_m) n l \hat{z}$

16. $V = 2\pi^2 R^2 f B \sin(2\pi ft + \pi/2)$

17. $\vec{H} = -\frac{\varepsilon_0 \omega V_0 d_1 s \cos(\omega t)}{2 \left( d_0 + d_1 \sin(\omega t) \right)^2} \hat{\phi}$

18. same as the last problem, but with $\omega$ replaced by $\varepsilon$

19. $\rho_{\text{free}} = 0; \; \vec{J}_{\text{free}} = -\frac{2\pi}{a} H_0 \sin\left( \frac{\pi x}{a} \right) \cos(kz - \omega t) \hat{y} ; \; \vec{J}_d = \frac{\varepsilon_0 \omega^2 a H_0}{\pi} \sin\left( \frac{\pi x}{a} \right) \cos(kz - \omega t) \hat{y}$