Wave information

1) General exponential dependence is  
   \[ e^{-i(k \cdot \mathbf{r} - \omega t)} \]

2) Direction of travel tells you what \( k \) components are present.

3) Direction of oscillation (i.e., longitudinal vs. transverse, and which specific transverse direction) tells you which \( \mathbf{u} \) components are present.

How to use Eqn 3.57 (a), (b), (c).

1) The Easy Way — If you know the specifics of \( u_x, u_y, u_z \), plug that in from the start.

Example 1 — (100) waves, longitudinal

Given \( \mathbf{k} = (k, 0, 0) \) and \( \mathbf{u} = [u_x, u_y, u_z] e^{i(k \cdot \mathbf{r} - \omega t)} \)

Eqn 3.57a becomes

\[
pw^2 \mathbf{u} = C_{11} k^2 \mathbf{u} \rightarrow \mathbf{u} = \frac{\mathbf{w}}{k} = \sqrt{\frac{C_{11}}{p}}
\]

Example 2 — (100) waves, transverse in the \((0, 1, 0)\) direction

Given \( \mathbf{k} = (k, 0, 0) \) and \( \mathbf{u} = (0, u_y, 0) \)

1. a. \( \mathbf{u} = u_y \mathbf{e}^{i(k \cdot \mathbf{r} - \omega t)} \)

Eqn 3.57a becomes

\[
pw^2 \mathbf{u} = C_{44} k^2 \mathbf{u} \rightarrow \frac{\mathbf{w}}{k} = \sqrt{\frac{C_{44}}{c}}
\]

Example 3 — (110) waves, longitudinal

Given \( \mathbf{k} = k \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) \) and \( \mathbf{u} = u \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) e^{i(k \cdot \mathbf{r} - \omega t)} \)

Eqn 3.57a becomes

\[
pw^2 \mathbf{u} = C_{11} \frac{k^2}{2} u + C_{44} \frac{k^2}{2} u + C_{12} + C_{44} \left( \frac{k^2}{2} u \right)
\]

\[
\rightarrow \frac{\mathbf{w}}{k} = \frac{1}{p} \left( \frac{C_{11} + C_{12} + 2C_{44}}{2} \right)
\]
2) The Hard Way - if you don't know all of the specifics, you'll need to solve simultaneous eqns in an eigenvalue-type notation.

Example 4: (110) waves, unknown direction in x-y plane

Given $k = \hat{k} \left( \frac{x^2 + y^2}{2} \right)$, $\mathbf{u} = (u_x, u_y)$

Then unknown.

But $u_z = 0$ is known.

Eqn 3.57a: $\rho \omega^2 \mathbf{u} = C_{11} \left( \frac{k^2}{2} \right) u_x + C_{12} \left( \frac{k^2}{2} \right) u_y + \left( C_{12} + C_{44} \right) \frac{1}{2} \frac{k^2}{2} u_y$

Or $\rho \omega^2 u_x = \frac{1}{2} k^2 \left( C_{11} + C_{44} \right) u_x + \frac{1}{2} k^2 \left( C_{12} + C_{44} \right) u_y$

Eqn 3.57b: $\rho \omega^2 u_y = \frac{1}{2} k^2 \left( C_{11} + C_{44} \right) u_y + \frac{1}{2} k^2 \left( C_{12} + C_{44} \right) u_x$

Combine in matrix eqns:

$$\begin{pmatrix} \rho \omega^2 & 0 \\ 0 & \rho \omega^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{2} k^2 \begin{pmatrix} C_{11} + C_{44} & C_{12} + C_{44} \\ C_{12} + C_{44} & C_{11} + C_{44} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

To have non-zero solns, $\det \begin{pmatrix} \rho \omega^2 & 0 \\ 0 & \rho \omega^2 \end{pmatrix} - \det \begin{pmatrix} \frac{1}{2} k^2 \left( C_{11} + C_{44} \right) & \frac{1}{2} \left( C_{12} + C_{44} \right) \\ \frac{1}{2} \left( C_{12} + C_{44} \right) & \frac{1}{2} \left( C_{11} + C_{44} \right) - \rho \omega^2 \end{pmatrix} = 0$

$$\frac{1}{2} k^2 \left( C_{11} + C_{44} \right) - \rho \omega^2 = \pm \frac{k^2}{2} \left( C_{12} + C_{44} \right)$$

$$\frac{u_x}{k} = \frac{1}{2} \sqrt{\frac{1}{2\rho} \left( C_{11} + 2C_{44} \right)}$$

Two solns: either

$$\frac{u_x}{k} = \sqrt{\frac{1}{2\rho} \left( C_{11} + 2C_{44} \right)}$$

Or

$$\frac{u_x}{k} = \sqrt{\frac{1}{2\rho} \left( C_{11} - C_{12} \right)}$$

To determine direction of oscillation for these velocities, plug that $\frac{u_x}{k}$ value back into Eqn 3.57 and solve for $\frac{u_y}{u_x}$ ratio. One turns out to be $\frac{u_y}{u_x}$, the other $\frac{u_y}{u_x}$.