Complex Numbers
- A horizontal

Euler $e^{i\theta} = \cos \theta + i \sin \theta$

Re-writing the other way
$Ae^{i\theta} = A \cos \theta + i A \sin \theta$

so $A \cos \theta = \text{Real} [Ae^{i\theta}]$

Similarly, find $A$ in this:
$5 \cos(\theta + 30^\circ) = 5e^{i(\theta + 30^\circ)}$

Phase notation:
- Real part assumed

When all terms in an equation have same

eigenvalues, we could have

worry about them if they all change

$e^{i\theta} \text{ assumed, I can do if all terms in each will have }$

\[
\begin{align*}
\text{How to add } & 7 \angle 30^\circ + 5 \angle 100^\circ \text{ ?} \\
& \frac{7}{\sqrt{2}} + 5 \frac{1}{\sqrt{5}} 
\end{align*}
\]
Complex Numbers Summary, by Dr Colton
Physics 471 – Optics

We will be using complex numbers as a tool for describing electromagnetic waves. *P&W* has a short section in Chapter 0 on the fundamentals of complex numbers, section 0.2, but here is my own summary.

Colton’s short complex number summary:

- A complex number *x* + *iy* can be written in rectangular or polar form, just like coordinates in the *x*-*y* plane.
  - The rectangular form is most useful for adding/subtracting complex numbers.
  - The polar form is most useful for multiplying/dividing complex numbers.
- The polar form (*A*, *θ*) can be expressed as a complex exponential *Ae*^*iθ*.
- For example, consider the complex number 3 + 4i:
  = (3, 4) in rectangular form,
  = (5, 53.13°) in polar form, and
  = 5e^i53.13° or 5e^i(0.9273 rad) in complex exponential form, since 53.13° = 0.9273 rad.
- The complex exponential form follows directly from Euler’s equation: *e*^*iθ* = cos*θ* + isin*θ*, and by looking at the *x*- and *y*-components of the polar coordinates.
- By the rules of exponents, when you multiply/divide two complex numbers in polar form, (*A*₁, *θ*₁) and (*A*₂, *θ*₂), you get:
  - multiply: *A*₁*e^i*θ*₁ * *A*₂*e^i*θ*₂ = *A*₁*A*₂*e^i(*θ*₁+*θ*₂) = (*A*₁*A*₂, *θ*₁+*θ*₂)
  - divide: *A*₁*e^i*θ*₁ / *A*₂*e^i*θ*₂ = (*A*₁/*A*₂)*e^i(*θ*₁-*θ*₂) = (*A*₁/*A*₂, *θ*₁-*θ*₂)
- I like to write the polar form using this notation: *A*∠*θ*. The “∠” symbol is read as, “at an angle of”. Thus you can write:
  = (3 + 4i) × (5 + 12i)
  = 5∠53.13° × 13∠67.38°
  = 65∠120.51° (since 65 = 5 × 13 and 120.51° = 53.13° + 67.38°)

Representing waves as complex numbers:

Suppose you have an electromagnetic wave traveling in the *z*-direction and oscillating in the *y*-direction. The equation for the wave would be this:

\[ \vec{E} = E₀ \hat{y} \cos(kz - ωt + φ) \]

It’s often helpful to represent that type of function with complex numbers, like this:

\[ \vec{E} = E₀ \hat{y} \cos(kz - ωt + φ) \quad \rightarrow \quad \vec{E} = E₀ \hat{y} e^{i(kz - ωt + φ)} \]

It’s understood that this is just a temporary mathematical substitution. If you want to know the real oscillation, you take the real part of the complex exponential, i.e. turn it back into a cosine.

\[ \rightarrow \vec{E} = E₀ \hat{y} \cdot e^{i(kz - ωt)} \]

Now \( \vec{E}_₀ \) is actually a complex number whose magnitude is \( E₀ \), the wave’s amplitude, and whose phase is \( φ \), the phase of the oscillating cosine wave. This type of trick will make the math much easier for some calculations we need to do.
Application: Number made up by class. Like this: 0.20

\[ \cos(3t + \_\_) + \_\_\_\_ \cos(3t + \_\_\_\_) \]

\[ = A \cos(3t + \phi) \]

What are A and \( \phi \)?

Taught in my 123 section... like adding vectors!

\[ \text{Re} \left[ A_1 e^{i(3t + \phi)} + A_2 e^{i(3t + \phi_2)} \right] \]

\[ \text{Re} \left[ e^{i3t} (A_1 L_1 + A_2 L_2) \right] \]

Add polar cards = add vectors.

\[ A L_0 \]

\[ \text{Re} \left[ e^{i3t} A e^{i\phi} \right] \]

\[ \text{Re} \left[ A e^{i(3t + \phi)} \right] \]

\[ A \cos(3t + \phi) \]

Test with Mathematica

Challenge: Work out this answer some other way!

Don't: points for first person

Often, on understand Re[], understand e^{i\theta}, even e^{i(\theta+\phi)}
Complex amplitudes

Ex. \[ E = 8^\text{v} \cos \left( kz - \omega t + 30^\circ \right) \]

\[ E_0 = \text{amplitude} = 8^\text{v} \]
complex amplitude = \[ 8 \angle 30^\circ \] \[ \text{incorporates phase into } E_0. \]

Note: regular amplitude is regular vector symbol \[ E_0 \]
complex amplitude is new vector symbol \[ E_0 \]
(Griffiths notation)
but polar does use \[ \angle \]

\[ E_0 \] is understood complex phase

polar coord. angle \[ \rightarrow \] complex phase.
E x B vanishes, a few quick notes

- If $\mathbf{B}$ also has a wave equation, identical when in free space (as mentioned).

- If you plug $E = E_0 \cos(kt-x\omega)$ into Faraday's law...

  $E_0 \approx i(2\pi/n)$

  $B \approx B_0 \omega/m$

  $i(k \cdot E_0) / E_0 \approx -B_0 \approx \frac{k}{n} E_0$

(a) same phase! 

(b) $B_0 = \frac{1}{\sin(\theta)} E_0 \cdot \hat{n}$

(c) $|B_0| = \frac{1}{z} |E_0|$

small!
Recall \( \nabla \cdot \mathbf{D} = \rho \) Gauss

\[ \mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{def!} \]

Before we used \( \mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E} \) to obtain

First eqn with \( \mathbf{v} = \frac{\mathbf{E}}{\sqrt{\varepsilon_0}} \) (if \( \mu = 1 \), i.e. nonmagnetic)

\[ \mathbf{v} = \frac{\mathbf{E}}{\sqrt{\varepsilon_0}} \]

But does \( \mathbf{D} \) always equal \( \varepsilon_0 \varepsilon_0 \mathbf{E} \)? Not necessarily

\[ \mathbf{P} \text{ must be related to } \mathbf{E} \text{ by } \mathbf{P} = \chi \mathbf{E} \]

\[ \chi \text{ is the susceptibility, } \mathbf{P} = \chi \mathbf{E} \]

(1) \( \chi \) could be matrix:

\[ \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \chi \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \]

\[ \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \] units

\[ \chi \text{ is usually diagonal, often } \chi = \begin{pmatrix} \chi_x & 0 & 0 \\ 0 & \chi_y & 0 \\ 0 & 0 & \chi_z \end{pmatrix} \]

Must have this if \( \chi \) nonisotropic

i.e. \( \chi \) in different directions than \( \mathbf{E} \)

(2) \( \chi \) could induce nonlinearity

\[ \mathbf{P} = \varepsilon_0 \left( \chi \mathbf{E} + \chi_2 \mathbf{E}^2 + \ldots \right) \]

(3) \( \chi \) can depend on frequency of light

(4) \( \chi \) could be complex - phase shift between \( \mathbf{E} + \mathbf{P} \)

Will worry about (3)(4) now, (1) later (Ch. 5) and (2) not at all in this class, e.g. Ph. 571