FORMULAS AND CHAPTER SUMMARIES
PHYSICS 220

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Contents

1 Formulas

PHYSICS 122: FORMULAS FROM THE TEXT,
THE LECTURES, AND THE ACCUMULATED WISDOM OF THE WORLD

Table of Physical Constants

$c = 2.9979 \times 10^8$ m/s	$\epsilon_0 = 8.8542 \times 10^{-12}$ F/m	$\mu_0 = 1.2566 \times 10^{-6}$ H/m
$e = 1.6022 \times 10^{-19}$ C	$m_e = 9.1094 \times 10^{-31}$ kg	$m_p = 1.6726 \times 10^{-27}$ kg
$1eV = 1.6022 \times 10^{-19}$ J	$\mu_B = 9.2740 \times 10^{-24}$ A $\cdot$ m^2	$k_e = 8.9876 \times 10^9$ N $\cdot$ m^2/C^2
$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2	$g = 9.80$ m/s^2	

Geometry

Circle: Circumference $= 2\pi r$ Area $= \pi r^2$ **Sphere**: Area $= 4\pi r^2$ Volume $= \frac{4}{3}\pi r^3$

SI prefixes

Factor	Prefix	Symbol	Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{-15}	femto	f	10^{-3}	milli	m	10^{3}	kilo	k
10^{-12}	pico	p	10^{-2}	centi	c	10^{6}	mega	M
10^{-9}	nano	n	10^{-1}	deci	d	10^{9}	giga	G
10^{-6}	micro	μ	10^{2}	hecto	h	10^{12}	tera	T

Chapters 23-24: Charge, Coulomb's Law, Electric Field, Gauss's Law

$F = \frac{q_1 q_2}{4\pi\epsilon_o r^2} = k_e \frac{q_1 q_2}{r^2}$, $\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \ldots$, $\mathbf{E} = \mathbf{F}/q_o$, $\mathbf{F} = q\mathbf{E}$

$\mathbf{E} = k_e \frac{q\mathbf{r}}{r^3}$, $\mathbf{E} = k_e \int \frac{dq\mathbf{r}}{r^3}$, $\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}$ $\epsilon_o \oint \mathbf{E} \cdot d\mathbf{A} = q$,

Sphere: $E = k_e \frac{q(r)}{r^2}$, Cylinder: $E = \frac{2k_e \lambda(r)}{r}$, Sheets: $E = \frac{\sigma}{2\epsilon_o}$ (insulator) or $\frac{\sigma}{\epsilon_o}$ (conductor)

Motion: $x = x_o + v_o t + \frac{1}{2}at^2$, $v = v_o + at$, $v^2 - v_0^2 = 2a(x - x_0)$,

Chapters 25-26: Potential (Voltage), Energy, Capacitance

$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$, $V = k_e \frac{q}{r}$, $V = k_e \int \frac{dq}{r}$, $V = k_e \sum \frac{q_i}{r_i}$

$U = k_e \frac{q_1 q_2}{r_{12}}$, $U = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \ldots \right)$, $U = qV$, $K = \frac{1}{2}mv^2$, $U + K = \text{const}$

$\mathbf{p} = q\mathbf{L}$, $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$, $U = -\mathbf{p} \cdot \mathbf{E}$

$Q = CV$, $C = \frac{\kappa \epsilon_o A}{d}$, $U = \frac{1}{2}CV^2$, $u = \frac{1}{2}\kappa\epsilon_o E^2$,

Series: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$, Parallel: $C_{eq} = C_1 + C_2 + C_3 + \ldots$

Chapters 27-28: Current, Resistance, Power, Emf, Circuits

$I = \pm \frac{dQ}{dt}$, $I = \int \mathbf{J} \cdot d\mathbf{A}$, $J = \frac{I}{A}$, $\mathbf{J} = nq\mathbf{v}_d$, $R = V/I$

$\rho = \frac{E}{J}$, $\sigma = \frac{1}{\rho}$, $R = \frac{\rho \ell}{A}$, $P = IV$, $P = I^2 R = V^2/R$

$\Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots = 0$, $I_1 + I_2 + I_3 + \ldots = 0$, $V_{ab} = V_b - V_a = \sum_a^b \Delta V$

Series: $R_{eq} = R_1 + R_2 + R_3 + \ldots$, Parallel: $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

$\mathcal{E} = R\frac{dQ}{dt} + \frac{Q}{C}$, Charging: $Q = C\mathcal{E}\left(1 - e^{-t/RC}\right)$, Discharging: $Q = Q_o e^{-t/RC}$

Chapters 29-30: Magnetic Field, Torque, Ampere's Law, Displacement Current, Magnetism

$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$, $\quad d\mathbf{F} = id\mathbf{s}\times\mathbf{B}$, $\quad \mathbf{F} = I\mathbf{L}\times\mathbf{B}$

$\mu = NiA$, $\quad \boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ $\quad \boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}$, $\quad U = -\boldsymbol{\mu}\cdot\mathbf{B}$

Circular Motion: $a = \frac{v^2}{r}$ $\quad r = \frac{mv}{qB}$, $\quad \omega = 2\pi\nu = \frac{qB}{m}$

$dB = \frac{\mu_o I \sin\theta ds}{4\pi r^2}$, $\quad d\mathbf{B} = \frac{\mu_o I d\mathbf{s}\times\mathbf{r}}{4\pi r^3}$, $\quad$ Ampere: $\oint \mathbf{B}\cdot d\mathbf{s} = \mu_o I$

Solenoid: $B = \mu_o ni = \frac{\mu_o Ni}{\ell}$, $\quad$ Wire: $B = \frac{\mu_o I}{2\pi r}$, $\quad$ Loop (center): $B = \frac{\mu_o I}{2r}$

Displacement Current: $I_d = \epsilon_o \frac{d\Phi_E}{dt}$, $\quad \mathbf{J}_d = \epsilon_o \frac{d\mathbf{E}}{dt}$, $\quad \oint \mathbf{B}\cdot d\mathbf{s} = \mu_o(I_d + I)$,

Magnetic Materials:

$\mathbf{M} = \frac{\boldsymbol{\mu}}{V}$, $\quad \mathbf{B} = \mu_o(\mathbf{H}+\mathbf{M})$, $\quad \mathbf{M} = \chi\mathbf{H}$, $\quad \mathbf{B} = \mu_m\mathbf{H}$, $\quad \mu_m = \mu_o(1+\chi)$, $\quad M = C\frac{B}{T}$

Chapter 31: Faraday's Law, Lenz's Law, Motional Emf, Maxwell's Equations

$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{A}$, $\quad \Phi_B = BA\cos\theta$, $\quad \mathcal{E} = -N\frac{d\Phi_B}{dt}$, $\quad \mathcal{E} = N\oint \mathbf{E}\cdot d\mathbf{s}$

$\oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$, $\quad \mathcal{E} = vBL$, $\quad \mathcal{E} = \mathbf{v}\times\mathbf{B}\cdot\mathbf{L}$

Maxwell's Equations:

$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q}{\epsilon_o}$, $\quad \oint \mathbf{B}\cdot d\mathbf{A} = 0$, $\quad \oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$, $\quad \oint \mathbf{B}\cdot d\mathbf{s} = \mu_o\epsilon_o\frac{d\Phi_E}{dt} + \mu_o I$

Chapters 32-33: Inductance, LC Oscillations, AC Circuits, Reactance, Transformers

$M = \frac{N_2\Phi_{21}}{I_1}$, $\quad \mathcal{E}_2 = -M\frac{dI_1}{dt}$, $\quad \mathcal{E}_1 = -M\frac{dI_2}{dt}$, $\quad L = \frac{N\Phi_B}{I}$ $\quad \mathcal{E} = -L\frac{dI}{dt}$

solenoid: $L = \mu_o n^2 \ell A$,

LR circuit: $L\frac{dI}{dt} + iR = \mathcal{E}$, $\quad$ (Turning on): $I = \frac{\mathcal{E}}{R}\left[1 - e^{-t/(L/R)}\right]$, $\quad$ (Turning off): $I = I_o e^{-t/(L/R)}$

$U = \frac{1}{2}LI^2$, $\quad u = \frac{B^2}{2\mu_o}$, $\quad U_E = \frac{Q^2}{2C}$, $\quad U_B = \frac{LI^2}{2}$,

LC circuit: $L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$, $\quad Q = Q_m\cos(\omega_o t + \phi)$, $\quad \omega_o = \frac{1}{\sqrt{LC}} = \frac{2\pi}{T}$,

LRC circuit: $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$, $\quad Q = Q_m e^{-Rt/2L}\cos(\omega_d t + \phi)$, $\quad \omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

AC circuit: $\mathcal{E} = V_m \sin\omega_e t$, $\quad I = I_m\sin(\omega_e t - \phi)$, $\quad X_C = \frac{1}{\omega_e C}$, $\quad X_L = \omega_e L$

$I_m = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V_m}{Z}$, $\quad \tan\phi = \frac{X_L - X_C}{R}$, $\quad Z = \sqrt{R^2 + (X_L - X_C)^2}$, $\quad P_{av} = \frac{1}{2}I_m^2 R = I_{rms}^2 R$

$P_{av} = \mathcal{E}_{rms}I_{rms}\cos\phi$, $\quad V_2 = V_1\frac{N_2}{N_1}$, $\quad V_1 I_1 = V_2 I_2$, $\quad I_{rms} = \frac{I_m}{\sqrt{2}}$, $\quad \mathcal{E}_{rms} = \frac{\mathcal{E}_m}{\sqrt{2}}$

Chapters 34, 16, 18: Waves, Electromagnetic Waves

Waves: $y = f(x \pm vt)$, $\quad k = \frac{2\pi}{\lambda}$, $\quad \omega = 2\pi\nu = \frac{2\pi}{T}$, $\quad y = y_m\sin(kx - \omega t + \phi)$, $v = \lambda\nu = \frac{\omega}{k}$

$c = \frac{1}{\sqrt{\mu_o\epsilon_o}}$, $\quad \mathbf{E}(x,t) = \mathbf{E}_m\sin(kx - \omega t)$, $\quad \mathbf{B}(x,t) = \mathbf{B}_m\sin(kx - \omega t)$, $\quad \frac{E}{B} = c$,

$\mathbf{S} = \frac{\text{Power}}{\text{Area}} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_o}$, $\quad \bar{S} = \frac{E_m B_m}{2\mu_o} = \frac{E_{rms}B_{rms}}{\mu_0}$, $\quad$ Pressure $= \frac{S}{c}$ (absorbed) or $\frac{2S}{c}$ (reflected) ,

Momentum $= \frac{U}{c}$ (absorbed) or $= \frac{2U}{c}$ (reflected),

A Few Useful Integrals

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$, $\quad \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}$, $\quad \int \frac{xdx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$, $\int \frac{xdx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$, $\quad \int \ln x dx = x\ln x - x$

2 Math Review

2.1 Vectors

We use *vectors* to describe any physical quantity which has both a magnitude and a direction. Common examples of such quantities are velocity, acceleration, force, the electric field, and the magnetic field. In the three-dimensional world in which we live, such vectors are specified by giving three components. For instance, we might write the velocity of a moving object in either of the forms

$$\mathbf{v} = (v_x, v_y, v_z)$$

or

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \quad .$$

We will usually write vectors using unit vectors, as in the second form above. The components v_x, v_y, and v_z tell how rapidly the object is making progress along each of the coordinate axes. The magnitude of the velocity is given by the familiar Pythagorean formula:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad .$$

2.2 Unit Vectors

A *unit vector* is a vector with magnitude one. We may easily convert any vector into a unit vector simply by dividing each component by the magnitude of the vector. For example, to make a unit vector, $\hat{n}$, pointing in the direction of the vector $\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, use the formula

$$\hat{n} = \frac{\mathbf{r}}{r} = \frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{14}} = 0.267\hat{\mathbf{i}} + 0.535\hat{\mathbf{j}} + 0.802\hat{\mathbf{k}} \quad .$$

2.3 Vector Operations

Vectors may be added, subtracted, multiplied by regular numbers (scalars), and multiplied with each other in two different ways. These operations will now be briefly reviewed.

Addition: Our world seems to be constructed in such a way that *vector addition* is often the correct thing to do. You are all familiar with the concept of adding two force vectors to obtain the total force. This might seem natural to you now, but it is a miracle that this simple operation describes what actually happens in the world. To add two vectors, we simply add their components, e.g.,

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} = (F_{1x} + F_{2x})\hat{\mathbf{i}} + (F_{1y} + F_{2y})\hat{\mathbf{j}} + (F_{1z} + F_{2z})\hat{\mathbf{k}}$$

Subtraction: Subtraction is, of course, just another form of addition, but it is worth mentioning here because we often want to find the position vector, $\mathbf{r}_{12}$, that starts at point

1, in three-dimensional space, and ends at point 2. For instance, the electric force exerted by a charge at point 1 on another charge at point 2 depends on $\mathbf{r}_{12}$. By drawing a picture on which points 1 and 2 are shown and where $\mathbf{r}_{12}$ is drawn, and by thinking about how you would move along each axis to get from point 1 to point 2, it is easy to see that the formula for $\mathbf{r}_{12}$ is given by

$$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \quad ,$$

where $\mathbf{r}_1$ is the position vector from the origin to point 1, and where $\mathbf{r}_2$ is the position vector from the origin to point 2.

Multiplication by a Scalar: To multiply a vector by a scalar, simply multiply each of its components by the scalar. For instance,

$$2\mathbf{v} = 2v_x\hat{\mathbf{i}} + 2v_y\hat{\mathbf{j}} + 2v_z\hat{\mathbf{k}} \quad .$$

This is often a very handy thing to do; for instance, in the example above we just doubled the velocity of an object. Multiplication by a scalar can also be used to reverse the direction of a vector: simply multiply by -1.

Scalar Product: The simplest way to multiply two vectors is called the *scalar product* (often called the *dot product*). The scalar product of two vectors, $\mathbf{A}$ and $\mathbf{B}$, is given by

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \quad .$$

What is this operation good for? We usually use the scalar product when what is important is not the whole magnitude of a vector, but only its magnitude along some direction. For instance, if a football player is running toward the goal line at an angle, his total speed is not nearly so important as his speed toward the goal line. If we set up a coordinate system in which the $\hat{\mathbf{i}}$ direction points from one goal post to the other, and if the football player's velocity is $\mathbf{v}$, then the speed at which he is approaching the goal line is simply $\mathbf{v} \cdot \hat{\mathbf{i}} = v_x$. Since our vectors are often written in unit vector form, it is helpful to memorize the rules for taking the scalar product of unit vectors:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

The scalar product is also useful if we want to know the angle between two vectors. Consider the two position vectors $\mathbf{r}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{r}_2 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. It is easy to compute their scalar product from their components:

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 = 9 \quad .$$

To find the angle between them, we use the other scalar product formula you might remember,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \Rightarrow \cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} \quad .$$

Since we already have the dot product, we only need the magnitudes of $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$r_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \quad ; \quad r_2 = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

The angle between them is then given by

$$\cos\theta = \frac{9}{\sqrt{3}\sqrt{29}} = 0.965 \quad ; \quad \theta = 15.2^\circ \quad .$$

Vector Product: The *vector product* (often called the *cross product*) is more complicated, and is used whenever something twists or circulates. Whenever you unscrew a lid or watch a toilet flush, think cross product. There are several ways to compute the vector product, but the most useful way for this course is to memorize the rules for the cross products of the unit vectors. And the easiest way to memorize these rules is to remember the following diagram:

$$\begin{array}{ccccc} & & \hat{\mathbf{i}} & & \\ & \nearrow & & \searrow & \\ \hat{\mathbf{k}} & & \longleftarrow & & \hat{\mathbf{j}} \end{array}$$

To take the cross product of two of these unit vectors, find them in the diagram above. If an arrow points from the first vector in the product to the second vector in the product, the answer is simply the third vector in the diagram. If an arrow points from the second vector in the product to the first, then the answer is the negative of the third vector in the diagram. And, best of all, if the two vectors in the product are the same, the answer is zero. Well, this sounds confusing, so here are some examples. Check each one against the rules and the diagram and make sure you understand how to use them; this will come in handy later.

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \quad ; \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \quad ; \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \quad .$$

There are nine possible products; six are given here. See if you can work out the other three.

2.4 Differentiation

We take a *derivative* (or *differentiate*) whenever we want to find the rate at which one quantity varies with respect to another. Common examples are velocity, which is the rate at which position changes with time, and acceleration, which is the rate at which velocity changes with time. We will encounter other important rates during our study of electricity and magnetism. To carry out calculations involving derivatives, you must know formulas for the derivatives of various functions. You are expected to memorize the following table:

$$\frac{dx^n}{dx} = nx^{n-1} \qquad \frac{d\ln x}{dx} = \frac{1}{x}$$

$$\frac{de^x}{dx} = e^x \qquad \frac{d\sin x}{dx} = \cos x \qquad \frac{d\cos x}{dx} = -\sin x$$

2.5 Integration

Integration is the main mathematical idea that makes this course so difficult. Most of you remember from calculus that integration finds areas under curves, or if you are really advanced, finds volumes of objects. We will use integration a lot in this course, but never to find areas or volumes (except in the second homework assignment). To survive, you will have to generalize your concept of what integration means. As explained in the long, detailed, and incredibly boring article in the student packet, integration is the process of adding up billions and billions of little bits of something to find the grand total. Little bits of mass can be added up to find the total mass; little bits of charge can be added up to find the total charge; little bits of force, electric field, voltage, current, magnetic field, etc., can all be added up via integration to find the total. Please sit down with pencil and paper and work through all of the examples in the article.

Integration problems can be incredibly difficult, and some of the homework problems will be challenging enough to force you to look some integrals up in the standard tables (CRC, Dwight, Pierce, Gradshteyn and Ryzhik). But for examinations you will only have to know the following integrals (and their cousins) from memory.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \qquad \int \frac{1}{x} \, dx = \ln x$$

$$\int e^x \, dx = e^x \qquad \int \sin x \, dx = -\cos x \qquad \int \cos x \, dx = \sin x$$

Fortunately, this table is simply the inverse of the differentiation table above. The cousins of these integrals are integrals like this:

$$\int (x+a)^n \, dx = \frac{(x+a)^{n+1}}{n+1} \qquad \int \frac{1}{(x+a)} \, dx = \ln (x+a)$$

$$\int e^{bx} \, dx = \frac{e^{bx}}{b} \qquad \int \sin bx \, dx = \frac{-\cos bx}{b} \qquad \int \cos b(x+a) \, dx = \frac{\sin b(x+a)}{b}$$

2.6 Densities

The crucial idea that connects integration with the physical world is the idea of *density*. For example, suppose we are trying to find the total electric force on a point charge, Q, due to a charged rod. (Assume that Q is located away from the rod, but along its axis so that we don't have to worry about vector integration.) Each of the charges on the rod is a different distance away from Q, and what's worse, the charge on the rod may not be distributed uniformly. It is difficult to see, at first, how to use our simple force formula for point charges,

$$F = k_e \frac{q_1 q_2}{r^2}$$

to deal with this situation. Well, if the formula only works for point charges, let's make sure we only have point charges by breaking the rod up into little tiny pieces, each approximately a point charge. We find the force exerted on Q by each little piece, then add all of the forces together to get the total. The addition of infinitely many infinitesimally small pieces is, of course, integration. This is conceptually quite simple, but when we try to carry it out we encounter a problem. The formula for the little bit of force exerted on Q by a little bit of charge dq is

$$dF = k_e \frac{Qdq}{r^2} \quad ,$$

where r is the distance between Q and dq. But what is the formula for dq?

This is where the idea of a density enters. If charges are distributed along a line or a curve, the important density is the *linear charge density*, denoted by the symbol λ. It tells us the charge per unit length at each point along the line, and gives us a relation between dq and the length of each tiny piece of the line, ds:

$$dq = \lambda ds \quad .$$

Similarly, if the charge is distributed over a surface, then the important density is the *area charge density*, the charge per unit area σ. In this case dq is given by

$$dq = \sigma dA \quad ,$$

where dA is the area of a tiny patch of the surface containing charge dq. Finally, if the charge is distributed throughout a volume, then we must know the *volume charge density*, the charge per unit volume ρ, which relates the volume of each tiny piece, dV, with the charge of each tiny piece, dq:

$$dq = \rho dV \quad .$$

All that remains is to find expressions for ds, dA, or dV using geometry, substitute for dq in the formula for dF, supply the proper limits of integration, and carry out the integration to obtain the total force. Several examples of how to use these densities can be found in the integration article in the student packet.

23 Charge and Electric Fields

BASIC CONCEPTS

23.1 Charge

Charge is the name we give to an interesting property of the particles that make up the atoms of the universe. We describe this property by saying that charge comes in two kinds, *positive* and *negative*, and that these two kinds of charge exert forces on each other according to the following rules.

1. Like charges repel.
2. Unlike charges attract.
3. The force gets stronger as the charges come closer.

The formula for the magnitude of the force between two point charges, q_1 and q_2, is given by *Coulomb's law*:

$$F = k_e \frac{q_1 q_2}{r^2} \quad ,$$

where r is the distance between the two charges. This formula is a very compact and precise way of expressing the rules given above. The rules for the multiplication of positive and negative numbers, which you probably learned in seventh grade, express rules 1 and 2: Positive times positive is positive, negative times negative is positive (likes repel); positive times negative is negative (unlikes attract). Rule 3 is expressed by the distance factor r^2 in the denominator, which causes F to become large as r gets small. The factor in front is simply a number which gives the correct force in Newtons when the charges are measured in Coulombs and the distance is measured in meters:

$$k_e = \frac{1}{4\pi\epsilon_o} = 8.99 \times 10^9 \mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}^2$$

Charge is "quantized", meaning that it comes in discrete units that cannot be split into smaller amounts. It is also apparently conserved, meaning that no matter how violently particles are smashed into each other and no matter how complicated the resulting reactions, the net amount of charge at the end is exactly the same as the amount before.

The matter in the universe is made up of very nearly equal amounts of positive and negative charge. The atoms that we, and our world, are made of consist of clouds of electrons, having negative charge, surrounding very small, dense nuclei made of protons and neutrons. The protons are positively charged, while the neutrons have no charge. (The protons in the nucleus repel each other, so it would explode unless there were a non-electrical force holding the protons together. Since this force must be stronger than electricity, physicists sensibly named it the strong force.) If an object has exactly equal amounts of positive and and negative charge, the object behaves as if it had no charge at all. For instance, a helium atom has two positive protons and two negatively charged electrons (for a net charge of zero). In spite of the fact that it contains charge, it behaves exactly like an uncharged object (except for polarization effects, discussed below).

APPLICATIONS

23.2 Insulators

An *insulator* is a material whose atoms **do** hold tightly to all of their electrons. Charge cannot flow freely through such materials, but they may be polarized (see below).

23.3 Conductors

A *conductor* is a material whose atoms **do not** hold tightly to all of their electrons. Instead, the electrons are shared among neighboring atoms, making a kind of free electron sea that can slosh around in the material.

23.4 Charging

Objects can be charged in several ways, and it is important to have a good mental picture of each of these processes.

Rubbing: When one insulator is rubbed with another insulator, at the places where they touch electrons can be exchanged. This can occur because even though the atoms in an insulator hold tightly to their electrons, some atoms attract electrons so strongly that they will take an extra electron away from a different atom. Hence, when differing atoms are placed in close contact with each other (by rubbing for example), electrons can be exchanged. The atoms that take on extra electrons form a region of negative charge, while those that give up electrons form a region of positive charge. Because the materials are insulators, these charged regions remain fixed; the charges are not free to move from place to place in these materials. Common examples of this process are: (i) When a plastic rod is rubbed with cat fur, the rod becomes negative while the cat fur becomes positive. (ii) When a glass rod is rubbed with silk or with a plastic sheet, the rod becomes positive while the silk or plastic sheet becomes negative.

Conduction: When an isolated conducting object is touched by a charged region on an insulator, or by another charged conductor, the conducting object receives the same kind of charge as on the object that touched it. To see why, suppose an insulator is charged negatively. When the insulator touches the conductor, some of the extra electrons on the insulator are attracted by the atoms in the conductor and jump into the conductor, leaving neutral atoms behind. The electrons that jump join the sea of free electrons, but because the conductor now has a net negative charge, they repel each other and try to get as far away from each other as possible. The electron sea comes **very quickly** into equilibrium with the excess electrons spread out over the surface of the conductor. Now suppose that the insulator is charged positively. When the insulator touches the conductor, the positive atoms (called ions) in the insulator, which have lost electrons, attract electrons from the sea in the conductor, becoming neutral atoms again. There is now a spot on the conductor that is positive, and free electrons are attracted to it, tending to fill it in. The end result of this attempt by the free electrons to fill in all of the positive spots is to leave a layer of positive charge spread out over the surface of the conductor. Even though the physical processes in these two cases are quite different, the final charge distributions on the conductors are identical, except for the sign of the charge. In other words, the distribution of positive charge on a positively charged conductor is the same as if it were the positive charges that were free to repel each other and spread out over the surface. It is impossible, by doing electrostatic experiments alone, to discover the sign of the free charge in conductors.

Induction: It is possible to charge a conductor negatively by using a positive rod, and to charge it positively by using a negative rod. We call this process *charging by induction*. The basic idea is to isolate the conductor so that electrons cannot leave or arrive from elsewhere. For example, imagine placing a charged rod, say negative, close to one end of an isolated conductor Some of the free electron sea inside is repelled away to the far end of the conductor, leaving positive charge at the end closest to the rod you brought up. Now touch the far end of the conductor with your finger, or a grounded wire, allowing the excess negative free electrons at that end to escape. Finally, remove your finger to trap the remaining positive charge on the conductor, and take the negative rod away. Voila! The conductor has now been charged positively with a negative rod.

23.5 Polarization

When a conductor is placed close to a charged object, the free electron sea responds either by moving closer to or further away from the charged object, depending on the sign of charge on the object. This shift of charge we call *polarization*. Insulators have no free electron sea, but a similar shift of charge takes place in these materials as well, and this shift is also called polarization. The reason for the shift in insulators is that each atom in the insulator is stretched by the charged object. To visualize this shift, imagine a positively charged object near an insulator. The electron cloud around each atom in the insulator is attracted toward the object while each nucleus is repelled. This means that each atom becomes a little bit positive on the side away from the positive object and a little bit negative on the side closer to the object. Within the body of the insulator this stretching has no effect, because for each slightly negative side of one atom, there is a corresponding slightly positive side of a neighboring atom to cancel it. The net effect is no net charge within the insulator. But on the surface of the insulator it is a different story. On the surface away from the positive charge, the slightly positive side of each stretched atom hangs out over the edge of the insulator with no neighboring atoms to cancel the positive charge. The surface far away from the positive charge is, then, charged slightly positive. Similarly, the surface close to the positive charge is charged negatively by the slightly negative sides of each stretched atom hanging out over the edge of the near surface. When an insulator is charged in this way, positive on one side and negative on the other, but with no net charge on the insulator, we say the insulator is polarized. Capacitors take advantage of this effect, as you will learn in Chapter 26.

23.6 Electric Field

We hardly ever use Coulomb's force formula; instead, we invent a vector quantity called the *electric field* from which we obtain electric forces when we need them. The definition of the electric field, **E**, is that it is the ratio between the electric force acting on a charge q and the charge itself:

$$\mathbf{E} = \frac{\mathbf{F}}{q} \quad .$$

The electric field is more than a formula, however; it has turned out to be a powerful concept for describing the world. We no longer believe that charged particles simply push and pull on each other. Instead, we believe that each charged particle carries with it an electric field, which in turn pushes or pulls on other charges.

If you want to know the direction of the electric field at some point, imagine placing a **positive** charge there. The direction of the force on the charge is the direction of **E**. And if you have a charge, q, and you want to know the force that will act on it at a place where the electric field is **E**, then use the formula

$$\mathbf{F} = q\mathbf{E} \quad .$$

Note that if q is negative, then the force points in the direction opposite to **E** because multiplying a vector by a negative sign reverses its direction.

23.7 Lines of Force

Lines of force or *electric field lines*, are imaginary lines formed by stepping along through space following the electric field arrows. Here are some useful rules and properties to help you draw these lines and tell what they mean.

1. Lines of force start at positive charges and end at negative charges.

2. The electric field is strong where the field lines are close together, and weak where they are far apart.

3. Lines of force cannot cross, **except** at points where $\mathbf{E} = \mathbf{0}$. At such points the lines **must** cross, usually with two lines coming in and two going out:

$$\begin{array}{ccc} & \downarrow & \\ \longleftarrow & + & \longrightarrow \\ & \uparrow & \end{array}$$

4. The number of lines connecting to a point charge is proportional to its charge. For instance, in a picture with charges of 1μC, 2μC, and 3μC, we could have 4 lines from the 1μC charge, 8 from the 2μC charge, and 12 from the 3μC charge.

APPLICATIONS

23.8 Point Charge

The electric field produced in the space around a point charge, q, is given by the formula

$$\mathbf{E} = k_e \frac{q\mathbf{r}}{r^3} \quad .$$

In this formula **r** is the position vector that points **from** the charge **to** the point where **E** is to be found. Sometimes we say that **r** points from the source to the observation point.

23.9 Particle Motion

Electric forces are just like all other forces; they cause particles to accelerate. The acceleration vector is related to the electric field vector by the formula

$$\mathbf{a} = \frac{q\mathbf{E}}{m} \quad ,$$

where m is the mass of the particle. In the case of a uniform electric field (meaning that $\mathbf{E}$ is constant in magnitude and points in the same direction at every point in space, producing straight, equally spaced field lines) the formulas for the motion of a particle are the same ones you learned in your first physics class:

$$\mathbf{v} = \mathbf{v}_o + \mathbf{a}t \quad ; \quad \mathbf{x} = \mathbf{x}_o + \mathbf{v}_o t + \frac{1}{2}\mathbf{a}t^2 \quad ; \quad v^2 - v_0^2 = 2a(x - x_0) \quad .$$

If the electric field is not uniform, as in the case of the electric field of a point charge, or in any case where the field lines curve, the motion of the particle is much harder to calculate.

24 Gauss's Law

BASIC CONCEPTS

24.1 Flux

Flux means "flow" and can be defined if we have a vector field and a surface. The definition of the flux of a vector field, say $\mathbf{E}$, through the surface A is

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A} \quad .$$

This integral is a *surface integral*, and you won't study integrals like these in detail until you take multidimensional calculus; but like any other integral, it is just a sum. In this case it is a sum over all of the little tiny areas, $d\mathbf{A}$, that make up the surface A. Each little patch of area is turned into an area vector by erecting on each one a flagpole unit vector. The integral says to take the dot product between $\mathbf{E}$ and $d\mathbf{A}$ at each little piece of area, and then to add them all up. The scalar product $\mathbf{E} \cdot d\mathbf{A}$ is a measure of how much of the vector field "flows", or passes through, the area $d\mathbf{A}$. The larger $\mathbf{E}$ is, the more flow there is; the larger the area is, the more flow there is; and the amount of flow through the surface varies as the orientation of the surface is changed (dot product). **In the case of the electric field, the flux Φ is proportional to the number of field lines that pass through the surface.**

The concept of flux is remarkably powerful; it underlies our understanding of many electromagnetic effects ranging from electric generators and magnetic fusion machines to radio, television, light, and X-rays. We will return to it in Chapters 30, 31, and 34.

24.2 Gauss's Law

Gauss's Law allows us to write, in a single equation, all of the rules about how charges make electric fields. This is the good news. The bad news is that to decode this equation to obtain all of these rules takes a PhD in applied mathematics. *Gauss's law* says: The electric flux through a **closed** surface is equal to the net charge enclosed by the surface, divided by ϵ_o:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_o} \quad .$$

The little circle on the integral sign is simply a reminder that the flux integral is to be taken over a closed surface. It does not mean that this is some fancy kind of integral. It is just a sum of little bits of flux, with one special feature: each of the little area vectors, $d\mathbf{A}$, points outward from the closed surface. This simple law is actually quite amazing. It says that no matter how distorted the electric field lines or how convoluted and twisted the surface, the flux integral through the closed surface is simply proportional to the net enclosed charge. This seems a little less amazing if we think about flux as the number of field lines that pass through a surface. If any surface whatever encloses a point charge, surely all of its field lines pass through the surface. But, you might object, what if the surface contains two point charges, one positive and one negative, of equal magnitude. The charge enclosed is then zero, but the field lines still loop out from the positive charge and return to the negative charge, passing through the surface on their way. Gauss's law still works, however, because of the way the flux integral counts field lines. If you think carefully about the flux product, $\mathbf{E} \cdot d\mathbf{A}$, you will see that outgoing field lines are counted as positive while incoming field lines are counted as negative. The flux through the surface enclosing two opposite charges is zero because every field line that goes out through the surface eventually comes back in on its way to the enclosed negative charge.

Gauss's law, in the hands of a trained professional like yourself, can be used to solve electric field problems that are quite difficult to solve by the methods of Chapter 23. After you take multidimensional calculus and advanced electromagnetism you too will have this power at your fingertips. In the meantime, all you need to know is that there is a compact way to write the laws governing electric fields, called Gauss's law, and that this law is useful for finding the electric field in a few cases where there is lots of symmetry. This law is the first of a set of four similar laws that we will encounter this semester. Together they constitute Maxwell's Equations, which describe nearly everything we know about electric and magnetic fields.

APPLICATIONS

24.3 A Charged Conductor

Gauss's law can be used to show that if an isolated conductor is charged, then after a brief period of time during which equilibrium is established and the charges come to rest, the excess charge will all be found on the surface of the conductor. Inside the conductor the electric field is exactly zero (if it weren't, the free electrons inside would be accelerated, and we would have to wait a little longer for equilibrium to be achieved). Just outside the conductor the electric field is perpendicular to the conductor surface and the magnitude of the surface electric field is related to the surface charge density by the formula

$$E = \frac{\sigma}{\epsilon_o} \quad .$$

24.4 Spherical Symmetry

If we have a spherically symmetric distribution of charge, then Gauss's law can be used to show that the electric field at any radius r from the center of symmetry is given by the following simple formula:

$$E(r) = \frac{q(r)}{4\pi\epsilon_o r^2} \quad = \quad k_e \frac{q(r)}{r^2} \quad ,$$

where $q(r)$ is the amount of charge within the sphere of radius r. The spherical shells of charge at distances greater than r produce zero net electric field at r.

24.5 Cylindrical Symmetry

If we have an infinitely long cylindrically symmetric distribution of charge, then Gauss's law can be used to show that the electric field at any radius r from the axis of symmetry is given by this simple formula:

$$E(r) = \frac{\lambda(r)}{2\pi\epsilon_o r} \quad = \quad 2k_e\frac{\lambda(r)}{r} \quad ,$$

where $\lambda(r)$ is the amount of charge **per unit length** within the cylinder of radius r. The cylindrical shells of charge at distances greater than r produce zero net electric field at r.

24.6 Planar Symmetry

There is a similar, but more complicated, result for charge distributions made up of uniformly charged infinite sheets, but it is probably better just to remember the formula for a thin sheet of charge with area charge density σ:

$$E = \frac{\sigma}{2\epsilon_o} \quad .$$

Note that the electric field points away from the sheet, for positive charge, and toward it for negative charge. Note also that the field does not change as the observation point is moved away from the sheet. Hence, a large sheet of uniformly distributed charge is the way to make a uniform electric field.

Be careful not to confuse the electric field formulas for the infinite sheet and for the charged conductor. The electric field at the surface of a conductor where the surface charge density is σ is twice as big as the field at the surface of an insulator with the same value of σ because conductors can't just be charged in one spot; the charge is free to move around, and it distributes itself over the entire surface. Hence, if you have charge density σ at some point on a conducting surface, there is a whole lot more charge distributed around the rest of it. The patch of charge near the surface makes a field of magnitude $\sigma/2\epsilon_o$ just like a similar patch on an insulator, but all the rest of the charge spread over the conductor surface contributes another field of magnitude $\sigma/2\epsilon_o$, making a total field of magnitude σ/ϵ_o.

25 Electric Potential (Voltage)

BASIC CONCEPTS

25.1 Electric Potential, or Voltage

Perhaps the best way to think about the meaning of *electric potential*, or *voltage*, is to make an analogy with gravity. When a mass is high above the ground we say that it has a lot of potential energy because if we drop it, it strikes the ground hard. We say that the potential energy due to height has been converted into kinetic energy. We can say the same things about electricity. If two positive charges are close together and we release them, they will repel each other and pick up speed, their electric potential energy having been converted into kinetic energy. In this analogy, the electrical quantity which corresponds to height is voltage.

Well, now we have to be more precise. The definition of the voltage difference between two points, a and b, in space is

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s} \quad ,$$

where this integral is a *path integral*, or *line integral*, starting at point a and ending at point b. The mental picture that goes with this path integral is this: Chop the path up into little segments of length ds and make them into vectors $d\mathbf{s}$ by making them point along the path from a to b. At each segment take the scalar product between $\mathbf{E}$ and $d\mathbf{s}$, then add all of these dot products together, starting at a and ending at b. For electric fields made by electric charges, the answer for the voltage difference is the same no matter what path is chosen. Applying this definition to the field of a point charge, q, and choosing the convention that the voltage is zero at infinity, we obtain the formula for the voltage produced at any point in space by a point charge:

$$V = \frac{k_e q}{r}$$

where r is the distance from the point charge to the point at which it is desired to know V. This formula says that the voltage is large and positive near positive charges and large and negative near negative charges. It can be used to find the potential produced by rods, rings, etc., by integrating, just as we did with electric fields in Chapter 23.

25.2 Equipotential Surfaces

An *equipotential surface* is a surface on which the potential, or voltage, is constant. Electric field lines are always perpendicular to these surfaces, and the electric field points from surfaces of high potential to surfaces of low potential. Suppose, for example, that a set of surfaces has been chosen so that their voltages are 5 V, 4 V, 3 V, 2 V, etc.. Then since the voltage difference between neighboring sheets is constant ($\Delta V = 1$ V) we can estimate the magnitude of the electric field between surfaces by the formula

$$E \approx \frac{\Delta V}{\ell}$$

where ℓ is the perpendicular distance between neighboring surfaces. (This formula is really just an approximate version of the path-integral definition of the voltage difference given above.) Note that this means that the electric field is strong where the equipotential surfaces are close together and weak where they are far apart.

25.3 Potential Energy

The *potential energy* of a charge, q, at a point in space where the voltage **due to all charges except q** is V, is given by the formula

$$U = qV \quad .$$

You can conceptually test for high and low potential energy by asking the question, "Does the charge want to go there or not?" If it is hard to get the test charge to go somewhere, then that is a place of high potential energy. If the test charge wants to go there, then it is a place of low potential energy. Notice that for a positive q, a place of high voltage is a place of high potential energy, but for a negative q, it is a place of low potential energy. *To avoid confusion, when testing for high and low voltage, always use a positive test charge.*

The total potential energy of a group of charges can be gotten by using the following formula for the potential energy of two point charges, q_1 and q_2, separated by a distance r_{12}:

$$U_{12} = k_e \frac{q_1 q_2}{r_{12}} \quad .$$

The potential energy of the group is simply the sum of the potential energies for all the pairs of charges in the group. For instance, for a group of four charges the total potential energy would be

$$U = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$$

APPLICATIONS

25.4 Conductors

If we wait until the charges on a conductor have stopped moving and have come into equilibrium, then the electric field lines at the surface of the conductor must be perpendicular to its surface. If this were not true, there would be a component of $\mathbf{E}$ along the conducting surface, and the free electrons would be accelerated. A sideways component of $\mathbf{E}$ at the surface means that we haven't waited long enough; wait a little longer and the charges will arrange themselves so that the electric field **is** perpendicular to the surface. But $\mathbf{E}$ is perpendicular to equipotential surfaces, so in *electrostatics* (no moving charges) the surface of a conductor is an equipotential surface. The interior of the conductor is at the same voltage as the surface because $\mathbf{E} = \mathbf{0}$ inside the conductor ($\int \mathbf{E} \cdot d\mathbf{s} = 0$ inside, so there can be no voltage differences between points in the conductor.) Hence, if part of a conductor is at a potential of 5 V, then the entire conductor is at 5 V. This makes electronic design possible. If the power supply on a computer produces 5 V, and if there is a conducting path starting at the power supply, going through the spaghetti of wires and metal trails on the back of circuit cards to a pin on a chip near the hard disk drive, then the voltage at the pin is also 5 V.

25.5 Ground

When we work with point charges, we usually use the convention that the voltage is zero at infinity. But when we work with circuits containing many charged wires, it is more convenient to have the zero of voltage somewhere nearby. The convention is to let the voltage be zero in the wet dirt of the earth. We say that an object has been *grounded* if it is connected by a conducting wire to the wet dirt, either by connection to water pipes, or by connection with the third hole in an electrical socket. Inside this small hole below the two slots in the socket is a copper wire which runs all through the building and eventually connects to the wet dirt of the earth, making it easy to ground objects anywhere in the building.

25.6 Particle Motion

If a charged particle moves without friction from point a to point b, the change in its velocity can be computed by energy conservation:

$$E_a = E_b \qquad \Rightarrow \qquad \frac{1}{2}mv_a^2 + U_a = \frac{1}{2}mv_b^2 + U_b \quad .$$

where $\frac{1}{2}mv^2$ is the kinetic energy of the particle. This formula is so useful for charged particles that a new unit of energy has been invented to go with it. One *electron volt* is defined to be the energy that one electronic charge would gain in being accelerated through a potential difference of 1 Volt, or

$$1 \text{ eV} = 1.602 \times 10^-19 \text{ J} \quad .$$

ADVANCED CONCEPT

25.7 Maxwell's Equations

If we choose a closed path along which to do the voltage line integral, we should get zero. Or in other words, the voltage difference between point a and point a is zero. This seems so obvious that it shouldn't even be worth mentioning, but let's write an equation for it anyway:

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0 \quad .$$

where the circle on the integral sign in this case means that the integral is to be taken along a closed path. Now we have two of Maxwell's Equations:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_o} \qquad ; \qquad \oint \mathbf{E} \cdot d\mathbf{s} = 0 \quad .$$

There are still two more equations to come, and even the two written here are incomplete. We will discover in Chapter 31 that the closed path voltage integral, which seems here so obviously to be zero, is in fact not zero at all if we include time-varying magnetic fields!

26 Capacitance

BASIC CONCEPTS

26.1 Capacitance

A *capacitor* is simply two pieces of conducting material insulated from each other. Normally, if one of the conductors is charged positively, the other is charged with an equal amount of negative charge. The positive conductor will be at a higher voltage than the negative conductor, and the charge is proportional to the voltage difference between them.

$$Q = CV \quad .$$

In this equation, and in all of the circuit relations we will use this semester, Q and V indicate magnitudes only. Note also that V is always the voltage difference. You will have to correctly handle negative signs yourself by drawing pictures and thinking carefully. The word capacitance refers to the capacity of the system to store charge. For the same voltage, a pair of conductors with a large value of C has a larger capacity to store charge than a pair with a small value of C.

26.2 Dielectrics

Dielectric Constant: If the gap between the conductors of a capacitor is filled with an insulator other than vacuum, the capacitance is increased because of the polarization of the insulator. It has been discovered that almost all insulators simply increase the capacitance by a factor, κ, called the *dielectric constant*:

$$C = \kappa C_{vacuum} \quad .$$

To change a formula that works in vacuum to one that works in a dielectric, simply replace ϵ_o everywhere by $\kappa\epsilon_o$. (Note: air is also a dielectric, but its dielectric constant is so close to 1 that we usually ignore it.)

Dielectric Strength: The *dielectric strength* of an insulator is the critical electric field beyond which the molecules in the dielectric stretch so much that one or more electrons are ripped free from the molecules. When this happens, the free electrons crash into other stretched molecules, knocking other electrons free; the resulting chain reaction is a spark that burns a hole through the dielectric. If this happens to the dielectric in a capacitor, the capacitor is usually ruined. The most spectacular example of this process is lightning, and the most annoying is when your little brother rubs his feet on the carpet and sneaks up from the rear.

APPLICATIONS

26.3 Capacitor Geometries

The capacitance of a system depends only on its shape and on the insulators it contains. In general, the capacitance is quite difficult to calculate, but if the geometry is symmetric, Gauss's law makes it possible to find formulas for C.

Parallel Plates: The simplest geometry is a pair of parallel plates, each with area A and separated from each other by a distance d which is small compared to the width of the plates. The capacitance of this system is

$$C = \frac{\epsilon_o A}{d}$$

Cylindrical Capacitor Another simple geometry is the coaxial cylinder in which an inner cylindrical conductor of length L and radius a is surrounded by an outer cylindrical conductor of length L and radius b. We assume that the length of the cylinder is much greater than its radius. (The round cable you use to connect a VCR to a TV set is an example of such a capacitor.) This system has capacitance

$$C = 2\pi\epsilon_o \frac{L}{\ln{(b/a)}} \quad = \quad \frac{L}{2k_e ln(b/a)} \quad .$$

Spherical Capacitor Finally, a spherical capacitor formed of two concentric spherical conducting shells, one with large radius b and the other with small radius a, has capacitance

$$C = 4\pi\epsilon_o \frac{ab}{b-a} \quad = \quad \frac{ab}{k_e(b-a)} \quad .$$

If the outer conductor is at infinity, we take the limit $b \to \infty$ to get the capacitance of an isolated sphere of radius a:

$$C = 4\pi\epsilon_o a \quad = \quad \frac{a}{k_e}$$

26.4 Series and Parallel

We say that two or more circuit elements are connected in *parallel* if the voltages across the elements are forced by the connections to be the same. We say that two or more elements are in *series* if, as we pass from one element to the next, the voltages add. When capacitors are connected in series, the charges on the capacitors are all equal. It is important to become proficient at using the voltage definitions of series and parallel, rather than relying on the appearance of the circuits, because circuit boards are rarely laid out with your convenience in mind.

Series: When capacitors are connected in series, their equivalent capacitance is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ...$$

Parallel: When capacitors are connected in parallel, their equivalent capacitance is

$$C_{eq} = C_1 + C_2 + C_3 + ...$$

If you must deal with a complicated combination of circuit elements that is neither series nor parallel, try to locate portions of the circuit that are simple series or parallel arrangements. Use the equivalent capacitance formulas to replace these portions with single equivalent elements, redraw the circuit, and look again for simple series and parallel arrangements. Keep at it until you reduce it to a single equivalent element. (This may not be possible for some arrangements.) If you need to find voltages and charges on circuit elements in the original combination, just start at the final single equivalent element, and work your way back through the intermediate circuit drawings until you reach the original circuit.

26.5 Capacitive Energy

When a capacitor is charged, the first little bit of charge requires no work to place it on the conductors because there is no charge present to repel it. But all of the following bits of charge are repelled by the charge placed on the conductors earlier, and hence energy must be expended to force them onto the conductors. The end result is that the charge on each conductor does not want to be there, and if an avenue of escape is provided, the charges will leave the conductors. Hence, a charged capacitor has electrical energy stored in it, and the amount of the energy, U, is given by the equivalent formulas

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} \quad .$$

26.6 Electric Energy Density

This is an action-at-a-distance picture of the stored energy: the charges repel each other during the charging process. The field picture of the energy is that the energy is stored in the space between the conductors where there is an electric field, E, present. Since the energy is thought of as being stored throughout a volume, it makes sense to speak of the volume density of this energy, or the energy-per-unit-volume. By rearranging the formula for U, given above, we discover that the formula for the volume *electric energy density* is

$$u_E = \frac{1}{2}\epsilon_o E^2 \quad .$$

In a dielectric the formula for the energy density is

$$u_E = \frac{\kappa}{2}\epsilon_o E^2 \quad .$$

Note that more energy is stored with a dielectric present than is stored in vacuum. The extra energy is stored in the stretching of the polarized atoms.

27 Current and Resistance

BASIC CONCEPTS

27.1 Current

Current is the rate at which charge passes a given point in a wire. If you were somehow able to count the charges as they went by in a wire carrying 1 ampere (1 A) of current, you would discover that every second an entire Coulomb of charge passes by. Formally we write

$$I = \frac{dQ}{dt} \quad .$$

The charge, Q, in this formula is the **net** charge passing a given point. For example, when a piece of matter is simply moved through space, an enormous amount of charge is in motion, but since equal amounts of positive and negative charge pass any given point, the net charge passing that point is zero, so the current is zero.

Inside the metal of a wire the free electrons are always in rapid thermal motion with speeds on the order of 10^5 m/s. This motion isn't current either because for every electron moving to the right, there is another one moving to the left, giving no net flow of charge. To set the charge in **net** motion requires that an electric field be applied, causing the entire rapidly moving cloud of free electrons to drift (rather slowly) through the metal.

27.2 Current Density

The current, I, is a coarse, average quantity that tells what is happening in an entire wire. If we want to describe in more detail how the charges move through a conductor, we use the concept of the *current density* $\mathbf{J}$:

$$\mathbf{J} = nq\mathbf{v}_d \quad ,$$

where n is the number of free charged particles per unit volume, q is their charge, and $\mathbf{v}_d$ is their average drift velocity. Since it is a vector quantity, it can indicate the direction of the current flow. Now for the bad news. This formula says that if free electrons are drifting to the right, then the resulting current flows to the left because the electrons are negative. It is difficult to keep this straight, so when we work with circuits, we usually just think of current as moving positive charges, even though we know that the electrons carry the current. The current density is an area density, I.e., $\mathbf{J}$ is the current per unit area. In fact, the current through an area A is simply the flux of the current density through that area:

$$I = \int \mathbf{J} \cdot d\mathbf{A} \quad .$$

Or, if the current density is uniform and flows straight into the area,

$$I = JA \quad .$$

27.3 Power

If a current I flows through a circuit element of any kind having voltage difference V across it, then the power either absorbed or produced by the element is given by

$$P = VI \quad .$$

If the current flows from high voltage to low voltage, this is the formula for the power absorbed, while if it flows from low voltage to high, it is the formula for the power produced by the element.

This formula, when combined with Ohm's law, can be used to obtain these two formulas for the power dissipated in a resistor:

$$P = I^2 R \qquad ; \qquad P = \frac{V^2}{R} \quad .$$

Students are often confused by these two formulas, because one of them says that if the resistance is increased the power increases, while the other says that the power decreases. The problem is that changing the resistance can also change the current or the voltage; to use these formulas successfully you must know whether it is the current or the voltage which is being held constant as the resistance is varied.

APPLICATIONS

27.4 AC and DC

We use the term *direct current*, or *DC*, to describe currents which are steady in time. We use the term *alternating current*, or *AC*, to describe currents which oscillate sinusoidally in time. The current flowing through the light bulb in a battery-powered flashlight is DC, while the current flowing through the light bulb in a lamp which is plugged into a wall socket is AC at a frequency of 60 Hz (in the USA). Direct currents are discussed in this chapter; alternating currents are discussed in Chapters 32 and 33.

27.5 Resistivity and Conductivity

In many conducting materials it is found that there is a simple relation between the applied electric field and the resulting current density:

$$\mathbf{E} = \rho \mathbf{J} \quad ,$$

where ρ is not the charge density, but is instead the *resistivity* of the material. Good conductors have low resistivities while bad conductors and insulators have high resistivities. If the resistivity is a constant, independent of the applied electric field, then we say that the material obeys Ohm's law.

The reciprocal of the resistivity is the *conductivity* σ:

$$\sigma = \frac{1}{\rho}$$

and the connection between the electric field and the current density can be written

$$\mathbf{J} = \sigma \mathbf{E} \quad .$$

27.6 Resistance and Ohm's law

For any piece of conducting material connected to a voltage difference, V, which drives through it a current I, we define the *resistance* of the conductor to be

$$R = \frac{V}{I} \quad .$$

Many of you are probably more familiar with this relation in the form

$$V = IR$$

and you are used to calling this *Ohm's law*. Everybody does this, but it is technically incorrect; this is really just the definition of resistance. **We say that the object obeys Ohm's law if R is a constant no matter what voltage is applied.** When a voltage difference is applied to a conductor there is a brief period of adjustment during which surface charges appear on the conductor in just the right places to produce an electric field inside the conductor to drive a current density in such a way that $\mathbf{E} = \rho\mathbf{J}$. This sounds like a very complicated process, and in general it is, but for a long skinny wire the end result of this arranging of surface charges is quite simple: if the current is steady in time, then $\mathbf{J}$ and $\mathbf{E}$ both point along the wire and are uniform throughout the volume of the wire. This makes it is easy to find the resistance of a conducting wire of length L, cross-sectional area A and resistivity ρ:

$$R = \frac{\rho L}{A} \quad = \quad \frac{L}{\sigma A} \quad .$$

But wait a minute, you object, we said before that $\mathbf{E} = \mathbf{0}$ in conductors. Yes, but only under electrostatic conditions. When a current flows in a conductor, there must be an electric field present to drive it. However, if the material is a very good conductor so that the resistivity, ρ, is very small, then even if a fairly large current density flows in the wire, the electric field can still be approximately zero. For example, if a current of 15 A flows in a copper wire of diameter 2 mm, the electric field inside the copper is only about 0.1 V/m. Hence, even when current is flowing, we often make the approximation that $\mathbf{E} \approx \mathbf{0}$ in good conductors.

27.7 Plumbing Analogy

It is difficult to keep straight the concepts of voltage, current, charge, resistance, etc., but for electrical circuits there is a helpful analogy with plumbing systems. Wires are, of course, like pipes; current is like the rate of water flow (gallons per minute); and resistance is like friction in the pipes. Now, here is the most helpful part of the analogy: voltage is like water pressure, and batteries and generators are like pumps. Like all analogies, however, it doesn't quite work. You may have seen movies where the hero rips a high voltage wire off the wall, frying the evil villain with a shower of sparks from the end of the wire. Sorry, but this can't happen. A water pump can pump water out of a hose into the air, but if an electrical pump (battery or generator) tried to pump electricity out of the end of a wire, nothing would happen because of the strong force of attraction between the free electrons and the positive ions in the wire. Water flow and electrical flow are only analogous if the water is confined to a closed system of pipes. With this warning in mind, let's do an example.

Water: Imagine connecting the outflow end of a powerful water pump to its inflow end with a hose full of sand. At two points along the hose there are flow meters, one near the outflow end of the pump, and one halfway along the hose. When the pump is turned on there is high pressure (pushing) at the outflow end and low pressure (suction) at the inflow end, but in spite of the large pressure difference, the two flow meters show that almost no water is flowing through the hose. In addition, the two flow meters give exactly the same small reading even though one of them is closer to the high pressure end. After thinking about this for a moment, you realize that this is as it should be. There is hardly any flow because of the large amount of friction between the water and the sand. And if the flow rate near the outflow were larger than the flow rate halfway down the hose, water would be building up in the hose. But it is nearly impossible to compress water, so the flow rates at the two places must be equal.

Electricity: Now, imagine connecting a high resistance wire from the positive end of a battery to its negative end. At two points along the wire there are ammeters, which measure current, one near the positive end of the battery and one halfway along the wire. When the connection is made there is high voltage at the positive end of the wire and low voltage at the negative end of the wire, but in spite of the large voltage difference, the two ammeters show that almost no electrical current is flowing through the wire. In addition, the two ammeters give exactly the same reading, even though one of them is closer to the high voltage end. After thinking about this for a moment, you realize that this is as it should be. There is very little current because of the large resistance of the wire. And if the electrical current near the positive end were larger than the current halfway down the wire, electrical charge would be building up in the wire. But the strong Coulomb repulsion of like charges makes it nearly impossible to concentrate electric charge, so the currents in the two places must be equal.

If you carefully compare these two paragraphs until you understand the analogy, it will be easier to understand how electrical circuits work.

28 Electromotive Force and Circuits

BASIC CONCEPT

28.1 Electromotive Force

Electromotive force, or *emf*, is the term we use to describe anything which behaves like an electrical pump. Batteries, generators, thermoelectric devices, solar cells, and piezoelectric crystals all do the same job in an electrical circuit: they pick conduction charges up at points of low potential energy and lift them up to high potential energy. If we imagine that current is positive charge in motion, then an emf pumps the current from low voltage up to high voltage. The symbol for emf is $\mathcal{E}$ and it has units of volts.

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28.2 Circuit Rules

It is helpful to translate the laws of attraction and repulsion into a few simple rules for analyzing circuits. The *loop rule* (or *loop theorem*), given below, is just a simplified version of the equation (see page 17 of this summary)

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0 \quad ,$$

while the *junction rule* simply says that large amounts of charge cannot build up at any point in the circuit because of the strong repulsion of like charges.

Loop Rule: Choose a closed path along the wires of a circuit. As you imagine traversing this path, note the voltage differences, ΔV, along the way, choosing the correct sign and magnitude for each one according to the rules given below. The sum of these voltage differences around the closed path must be zero:

$$\sum \Delta V = 0 \quad .$$

Junction Rule: At a junction between three or more wires, the sum of the incoming currents must equal the sum of the outgoing currents, or "what goes in must come out."

Emf Rule: If an emf is traversed in the direction in which it tends to pump charge, the voltage change is $\Delta V = \mathcal{E}$. If traversed opposite to its pumping direction, $\Delta V = -\mathcal{E}$.

Resistor Rule: If a resistor is traversed in the direction of the current passing through it, $\Delta V = -IR$. If traversed opposite to the current, $\Delta V = IR$.

Capacitor Rule: If a capacitor is traversed from the negative plate to the positive plate, $\Delta V = Q/C$. If traversed from positive to negative, $\Delta V = -Q/C$.

Relation of capacitor Q to I: If the direction of the current I is away from a positive capacitor plate having charge Q, then

$$I = -\frac{dQ}{dt} \quad .$$

If the current flows onto a positive capacitor plate, then

$$I = +\frac{dQ}{dt} \quad .$$

To use these rules to analyze a multiloop circuit, first locate all of the junctions in the circuit. Choose all but one of the junctions to include in the analysis. Then assign unknown current variables and directions to the branches between all of the junctions, and choose as many independent loops as required to get the same number of combined junction and loop equations as unknowns. Finally, solve the resulting set of simultaneous equations.

28.3 Voltage Differences

The loop rules may also be used to calculate the voltage difference between two points in a circuit. If V_a is the voltage at point a and V_b is the voltage at point b, simply choose a path that leads from a to b and evaluate all of the voltage differences ΔV along the way. Then use the formula

$$V_a + \sum_a^b \Delta V = V_b \quad .$$

28.4 Series and Parallel

We say that two or more circuit elements are connected in *parallel* if the voltages across the elements are forced by the connections to be the same. We say that two or more elements are in *series* if, as we pass from one element to the next, the voltages add. When resistors are connected in series, the currents through them are equal.

Series: When resistors are connected in series, their equivalent resistance is given by

$$R_{eq} = R_1 + R_2 + R_3 + ...$$

Parallel: When resistors are connected in parallel, their equivalent resistance is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...$$

28.5 Batteries

A *battery* is a combination of various chemicals and metal plates that generates an electromotive force. The magnitude of its emf is determined by the chemical reactions that take place inside it, no matter how old or worn out it is. For instance, even a dead 1.5 V battery generates an emf of 1.5 V. What, then, makes a battery go dead? The simplest way to describe what has happened is to say that its *internal resistance* has become large. The chemicals inside the battery must pass electricity easily or the emf will not be able to drive current in a circuit. A new battery has been carefully designed and manufactured to have low resistance, but as it is used the chemical reactions inside cause the cell to be less efficient at conducting electricity. Finally, when the battery has developed such a large resistance that its emf can't drive much current, we say that the battery is dead.

Because a battery combines an emf with a resistor, the voltage difference between the terminals of the battery, or its *terminal voltage*, is not determined by the emf alone. If current I flows through a battery with emf $\mathcal{E}$ and internal resistance r, then the terminal voltage of the battery is given by

$$V_{terminal} = \mathcal{E} - Ir \quad .$$

28.6 Measuring Instruments

Voltmeter: A *voltmeter* measures the voltage difference between two points in a circuit. The two wires of the meter connect it in parallel with the desired section of the circuit. A perfect voltmeter would have **infinite resistance** so that the circuit conditions would not have to change to supply current to the branch of the circuit containing the voltmeter.

Ammeter: An *ammeter* measures the current through a branch of a circuit. The meter is placed in series with the other circuit elements in the branch. A perfect ammeter would have **zero resistance** so that it would not alter the currents in the circuit.

28.7 Grounding a Circuit

It is permissible to connect a *ground* wire to any desired point in an isolated circuit. This simply defines the place in the circuit where the voltage is zero. You might think that since the ground wire goes to the wet dirt, the emf in the circuit would pump charge out the ground wire, but it doesn't. Any charges that happened to leave down the ground wire would be pulled back by the opposite charge they left behind. However, if two ground wires are connected to two different parts of the circuit, the circuit will be changed drastically.

28.8 RC Circuit

Discharging: If a charged capacitor, C, is allowed to *discharge* through a resistor, R, the initial large voltage difference on the capacitor causes the current to be fairly large at first. But as the charge drains off the capacitor, the voltage drops, the current weakens, and eventually the current becomes so small that it is of no further practical interest. (The plumbing analog of this process is the flushing of a toilet.) The loop theorem applied to this circuit gives the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad .$$

The solution of this equation corresponding to initial charge Q_o on the capacitor is

$$Q(t) = Q_o e^{-t/RC} \quad ,$$

The quantity RC is called the *capacitive time constant* or the *RC time*. Note that this means that an ohm multiplied by a farad is a second. The units in electromagnetism are truly hopeless, so you just have to memorize things like "RC is a time." This characteristic time is the time for the charge on the capacitor to drop from Q_o to $e^{-1}Q_o \approx 0.37Q_o$.

Charging: If an uncharged capacitor, C, is connected in series with an emf, $\mathcal{E}$, and a resistor, R, then initially the capacitor *charges* as rapidly as the resistor allows, since there is nothing on the capacitor to repel the incoming charges. Later on the charging process slows as the charge on the capacitor repels further arrivals. (The plumbing analog of this process is the filling of the toilet tank after a flush.) The loop theorem for this circuit gives the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E} \quad .$$

The solution of this equation corresponding to no initial charge on the capacitor is

$$Q(t) = C\mathcal{E}(1 - e^{-t/RC}) \quad .$$

In either the charging case or the discharging case, the current in the circuit can be obtained either from

$$I = |\frac{dQ}{dt}|$$

or by solving for I from the loop theorem:

$$\text{Discharging}: \quad -IR + \frac{Q}{C} = 0 \qquad ; \qquad \text{Charging}: \quad iR + \frac{Q}{C} = \mathcal{E} \quad .$$

29 What Magnetic Fields Do

BASIC CONCEPTS

29.1 Magnetic Field

Just as we associate an electric field with charges, we associate a *magnetic field* with magnets. At first, scientists tried to make a theory of magnetism similar to our theory of electricity. This seemed natural, since magnets attract and repel each other through empty space just like electric charges do. To approach magnetism in this way, we have to take the magnet as our basic object, with the *north pole* of the magnet analogous to positive charge, and the *south pole* of the magnet analogous to negative charge. The north pole is defined as the end of the magnet that points north; since unlike poles attract, this means that the north geographic pole of the earth is a south magnet pole. Magnetic field lines exit from north poles and enter at south poles, so the earth's magnetic field comes out of Antarctica and goes back into the earth in the Arctic.

A theory along these lines worked a little bit, but as experiments became more sophisticated, it became clear that it had some failings. One problem is that it turns out to be impossible to isolate the magnetic poles from each other. Imagine cutting a bar magnet in half. You might think you would have separate north and south poles, but instead you would have two bar magnets, each with a north and a south pole. Well, maybe you haven't cut it up fine enough yet. Break it down into individual atoms; but again, each atom has a north and a south pole. OK, you can do even better by breaking each atom up into protons, neutrons, and electrons; again, each particle has a north and a south pole. This is very frustrating, and suggests that north and south poles are not fundamental concepts like positive and negative charges. Another problem is that it was discovered that magnets not only push on other magnets, but that they also push on moving electric charges. This effect turned out to be so much easier to study than the interaction between magnets that we now use it to define what we mean by magnetic field. So even though we are studying magnetism, we will forget all about magnets now until we get to Chapter 30. The plan is to study what magnetic fields do in Chapter 29, then to study how they are produced in Chapter 30.

During the early 1800's it was discovered that magnets push on moving electric charges, and the modern definition of $\vec{B}$ depends on this effect. We say that the magnetic field at some point is $\mathbf{B}$ if the force exerted on a charge q at that point, moving at velocity $\mathbf{v}$, is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad .$$

This defining relation is analogous to the relation $\mathbf{F} = q\mathbf{E}$ for electric fields, but because of the cross product it is more difficult to use.

APPLICATIONS

29.2 Particle Motion

Two important properties of the magnetic force law, given above, determine the motion of a particle in a uniform magnetic field. (1) The force has no component in the direction of $\mathbf{B}$. This means that the component of a particle's velocity along the magnetic field is unchanged by the magnetic field. (2) The force is always perpendicular to the particle velocity, $\mathbf{v}$. This means that in a uniform magnetic field the particle motion will be circular, except for its uniform motion along the direction of $\mathbf{B}$. The particles thus travel in spiral paths along and around the magnetic field lines. If the speed of the particle in the directon perpendicular to the magnetic field is $v_{\perp}$, then the radius of the circular motion in the plane perpendicular to $\mathbf{B}$ is given by

$$r = \frac{mv_{\perp}}{qB} \quad ,$$

while the *period* T, *frequency* f, and *angular frequency* ω, of the circular motion are given by

$$T = \frac{2\pi m}{qB} \quad ; \quad f = \frac{1}{T} = \frac{qB}{2\pi m} \quad ; \quad \omega = \frac{2\pi}{T} = \frac{qB}{m} \quad .$$

Note that none of these formulas depends on the speed of the particle. This fact is routinely utilized in particle accelerators like the cyclotron and the synchrotron.

29.3 Magnetic Forces on Currents

Since magnetic fields push on moving charges, and since current is charge in motion, magnetic fields must push on currents. The most convenient way to write the formula for the force on a current is to write the formula for the small force, $d\mathbf{F}$, on a small segment of current carrying wire, of length ds, and current I. If we turn the small length into a vector $d\mathbf{s}$ by having it point in the direction of the current flow, then the formula for $d\mathbf{F}$ is

$$d\mathbf{F} = I d\mathbf{s} \times \mathbf{B} \quad .$$

To obtain the total force on an entire circuit, or on just a portion of the circuit, this formula must be integrated over all the segments $d\mathbf{s}$ of the circuit. If the magnetic field is uniform and the segment is straight, the integral is simple and gives

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B} \quad .$$

29.4 Torque on Current Loops and Magnetic Dipoles

If a current-carrying loop of wire is placed in a uniform magnetic field, the net force on the loop is zero. The magnetic field may, however, exert a *torque* (or twist) on the loop. The general formula for the torque on an object is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \quad ,$$

where $\mathbf{r}$ is the vector that points from the center of rotation to the point where the force $\mathbf{F}$ is applied. If this torque formula is integrated around a loop of current to find the net torque on the loop due to the uniform magnetic field, the result is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} \quad ,$$

where the quantity $\boldsymbol{\mu}$ is called the *magnetic dipole moment* of the loop, and is defined as follows. The magnitude of the dipole moment is

$$\mu = NIA \quad ,$$

where I is the current flowing in the wire, and where A is the area of the loop (assuming that the loop is flat), and where N is the number of turns of wire in the loop. The direction of $\boldsymbol{\mu}$ is determined by using the right-hand rule on the direction of current flow: curl your fingers in the direction of the current, and your thumb points in the direction of $\boldsymbol{\mu}$. We could also write

$$\boldsymbol{\mu} = NI\mathbf{A} \quad ,$$

where $\mathbf{A}$ is the area vector for the area bounded by the loop. Its magnitude is the area and its direction is perpendicular to the surface, just as we defined it when we discussed flux. The ambiguity in the direction of this vector is resolved by using the right-hand rule on the current, as discussed above.

The torque formula above shows that the loop will be in equilibrium (zero torque) only if $\boldsymbol{\mu}$ is either parallel with $\mathbf{B}$ or opposite to $\mathbf{B}$. The opposite position is unstable (like trying to balance a pencil point-down on your finger), so a free current loop tries to twist until its dipole moment vector is parallel with the magnetic field.

To see what happens to a current loop in a non-uniform magnetic field, we need another formula. The formula for the potential energy of a current loop in a magnetic field, uniform or nonuniform, is

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \quad .$$

Note that the potential energy will be lowest (most negative) when $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$, so the aligned position preferred by the loop is the position of lowest potential energy, as expected. Suppose, now, that the loop has its dipole moment aligned with $\mathbf{B}$, but that $\mathbf{B}$ is non-uniform. Does a force act on the loop, and if so, in what direction does it point? We decide this question, in the usual way, by thinking about the potential energy. Objects tend toward positions of lower potential energy; since the potential energy is already negative, lower potential energy would have to mean larger magnitudes of negative potential energy. This can be achieved if the loop moves to places where $\mathbf{B}$ is larger.

We may now see what a free current loop will do in a general magnetic field. First, it will tend to align itself with the magnetic field, and then it will be attracted to the place where the magnetic field is strongest. If you have played with magnets on a table, you may have noticed that this is exactly what they do. If one magnet is moved close to a second one, the free magnet will flip around, then rush toward the strong magnetic field at one of the poles of the first one.

Now we have a connection between electricity and magnets: magnets behave just like current loops. In fact, the dipole moment vector of a bar magnet points from its south end toward its north end, and the torque and potential energy formulas given above apply to bar magnets as well as to current loops. In Chapter 30 we will discover that this similarity in behavior is caused by strong atomic currents in bar magnets.

29.5 Hall Effect

The Hall Effect is simply an application of the total force law for charged particles:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad .$$

If a magnetic field is applied perpendicular to a strip of metal which is carrying a current, then the electric field in the metal points in the direction of current flow, and by paying close attention to the signs of q and $\mathbf{v}$ in the formula above, it is easy to discover that the magnetic field would push positive charge carriers toward one edge of the strip and negative charge carriers toward the other. This makes it possible to tell whether a conducting strip has positive or negative charge carriers by measuring the electrostatic voltage between opposite edges of the strip. There is no sense in trying to memorize special cases; just think carefully about current, electric field, and the motion of the charge carriers and use the total force law.

30 How Magnetic Fields are Produced

BASIC CONCEPTS

30.1 Ampere's Law

Ampere's law allows us to write down a single equation that describes all of the ways that electric current can produce a magnetic field. But, just as with Gauss's law, this single equation is very difficult to solve. Ampere's law says: The path integral of $\mathbf{B} \cdot d\mathbf{s}$ around any (imaginary) closed path is equal to the current enclosed by the path, multiplied by μ_o:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I_{enclosed} \quad .$$

The little circle on the integral sign reminds us that the integral must be taken around a *closed* path. To decide whether a particular current is enclosed by the path or not, imagine stretching a thin membrane over the path, like a drumhead, so that the edge of the membrane is along the path If a current pierces the membrane, it is an enclosed current. Currents that do not pierce the membrane are left out of Ampere's law, just as charges outside of the closed surface were left out of Gauss's law. Like the charges in Gauss's law, these enclosed currents are either positive or negative. Here is the rule for deciding whether an enclosed current is positive or negative. Curl the fingers of your right hand in the direction of integration around the path. If a current pierces the membrane stretched across the loop in the direction of your thumb, then it is a positive enclosed current. If a current pierces the membrane in the opposite direction, it is negative.

When we discussed Gauss's law, we noted that the law was true no matter how distorted the surface or how complicated the electric field. Similarly, Ampere's law is always true, no matter how distorted the path or how complicated the magnetic field. In most cases, however, even though Ampere's law is true, it is useless because it is impossible to perform the path integral. In a few special, symmetric situations, however, it is easy to perform the path integral and we can obtain formulas for B that would be quite difficult to derive with the Biot-Savart law.

30.2 Gauss's Law for Magnetic Fields

In spite of our best efforts, we have been unable to find magnetic charge in our part of the universe. Hence, magnetic field lines cannot start or stop anywhere but can only circulate, like the lines surrounding a current-carrying wire. This makes Gauss's law for magnetic fields quite simple:

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad .$$

30.3 Biot-Savart Law

The magnetic equivalent of Coulomb's law is the *Biot-Savart law* for the magnetic field produced by a short segment of wire, $d\mathbf{s}$, carrying current I:

$$d\mathbf{B} = \frac{\mu_o I}{4\pi} \frac{d\mathbf{s} \times \mathbf{r}}{r^3} \quad ,$$

where the direction of $d\mathbf{s}$ is in the direction of the current and where the vector $\mathbf{r}$ points from the short segment of current to the observation point where we are to compute the magnetic field. Since current must flow in a circuit, integration is always required to find the total magnetic field at any point. The constant μ_o is chosen so that when the current is in amps and the distances are in meters, the magnetic field is correctly given in units of tesla. Its value in our SI units is **exactly**

$$\mu_o = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} = 1.26 \times 10^{-6}\ \mathrm{T \cdot m/A} \quad .$$

A quick comparison of this value with the Biot-Savart law probably makes you wonder what role 4π is supposed to play here. It plays the same role it did in Coulomb's law: it was required in Coulomb's law so that Gauss's law wouldn't have a 4π, and it is required in the Biot-Savart law so that Ampere's law won't have one either.

There are two simple cases where the magnetic field integrations are easy to carry out, and fortunately they are in geometries that are of practical use. We use the formula for the magnetic field of an infinitely long wire whenever we want to estimate the field near a segment of wire, and we use the formula for the magnetic field at the center of a circular loop of wire whenever we want to estimate the magnetic field near the center of any loop of wire.

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Infinitely Long Wire: The magnetic field at a point a distance r from an infinitely long wire carrying current I has magnitude

$$B = \frac{\mu_o I}{2\pi r}$$

and its direction is given by a *right-hand rule*: point the thumb of your right hand in the direction of the current, and your fingers indicate the direction of the circular magnetic field lines around the wire.

Circular Loop: The magnetic field *at the center* of a circular loop of current-carrying wire of radius R has magnitude

$$B = \frac{\mu_o I}{2R}$$

and its direction is given by another *right-hand rule*: curl the fingers of your right hand in the direction of the current flow, and your thumb points in the direction of the magnetic field inside the loop.

Long Thick Wire: Imagine a very long wire of radius a carrying current I distributed symmetrically so that the current density, J, is only a function of distance r from the center of the wire. Ampere's law can be used to find the magnetic field at any radius r. Outside the wire, where $r \geq a$, we have

$$B = \frac{\mu_o I}{2\pi r} \quad ,$$

just as if all the current were concentrated at the center of the wire. Inside the wire, where $r < a$, we have

$$B = \frac{\mu_o I(r)}{2\pi r} \quad ,$$

where $I(r)$ is the current flowing through the disk of radius r inside the wire; the current outside this disk contributes nothing to the magnetic field at r. Note that this is analogous to the result for symmetric electric fields, discussed in Chapter 24.

Long Solenoid: Imagine a long solenoid of length L with N turns of wire wrapped evenly along its length. Ampere's law can be used to show that the magnetic field inside the solenoid is uniform throughout the volume of the solenoid (except near the ends where the magnetic field becomes weak) and is given by

$$B = \mu_o \frac{N}{L} I = \mu_o n I \quad ,$$

where $n = N/L$.

Toroid: Imagine a toroid consisting of N evenly spaced turns of wire carrying current I. (Imagine winding wire onto a bagel, with the wire coming up through the hole, out around the outside, then up through the hole again, etc..) Ampere's law can be used to show that the magnetic field within the volume enclosed by the toroid is given by

$$B = \frac{\mu_o N I}{2\pi R} \quad ,$$

where R is the distance from the z-axis in cylindrical coordinates, with the z-axis pointing straight up through the hole in the center of the bagel.

30.4 Forces Between Currents

The field picture of the interaction of two electric charges is that each charge produces an electric field that pushes or pulls on the other charge. We have a similar picture of the interaction between two currents: the current in each wire produces a magnetic field that pushes or pulls on the other current. If two neighboring wires carry currents in the same direction, they attract each other, while if they carry currents in opposite directions, they repel each other. For example, the two wires in the power cord of a table lamp exert magnetic forces on each other, and since the AC current oscillates, the forces oscillate. These oscillating forces are the source of the hum that you hear in many electrical appliances. If you need to find the magnitude of the force between two long parallel wires, simply compute the magnetic field produced by one wire at the location of the other, then use $F = ILB$ from Chapter 29 to get the force.

Current-carrying loops also exert forces on each other, and it is easy to find the directions of these forces if we use the idea of the dipole moment to convert the loops into magnets. Once the right-hand rule has been used to find the directions of the dipole moments of each loop, just think of the loops as magnets; usually this makes it possible to decide whether the loops attract, repel, or twist each other. Just remember that the dipole moment vector of a magnet points from its south pole to its north pole when converting loops into equivalent magnets.

30.5 Displacement Current

At about the time of the American Civil War, James Clerk Maxwell made an attempt to combine the best mathematics of his day with all of the experimental work on electricity and magnetism from the preceding hundred years. As he did so, he was mystified by Faraday's idea that the stored energy in a capacitor was stored in the electric field between the plates. Was the energy density formula,

$$u_E = \frac{\epsilon_o}{2}E^2$$

just a formula, or was the energy somehow really stored in space? As he thought about this formula, he realized that in a dielectric it was possible to see how the energy could be stored: it was stored in the stretching of the atoms of the material. The larger the electric field, the more the atoms were stretched, and when the electric field was removed, the atoms snapped back to their original state, giving up the energy that was stored in them. Taking this as a hint, Maxwell made the hypothesis that the vacuum was not really empty at all, but was instead filled with atoms of a very fine and insensible material which he called the *ether*. When electric energy was stored in space, Maxwell took this to mean that the atoms of the ether became stretched, just like the atoms in paper or oil.

Once he came to believe in this picture, he was led to the following brilliant insight: if ether atoms become stretched when an electric field is applied, then when the electric field is changing in time, there must be a current in the ether. This must be so because when an atom becomes more stretched by the increasing electric field, its positive charge moves in the direction of the applied electric field while its negative charge moves in the opposite direction. But this means that both moving charges contribute to current flow in the direction of the applied electric field. This current, Maxwell realized, must produce magnetic field, and hence should be added to the conduction current in Ampere's law. He called this current *displacement current* and found the following formula for the displacement current flowing through a surface:

$$I_d = \epsilon_o \frac{d\Phi_E}{dt} \qquad ; \qquad \Phi_E = \int \mathbf{E} \cdot d\mathbf{A} \quad ,$$

where Φ_E is our old friend, the electric flux through the surface. This formula can be used to find the total amount of displacement current flowing through a surface, but it doesn't indicate its direction. The direction is given by the following formula for the *displacement current density* $\mathbf{J}_d$:

$$\mathbf{J}_d = \epsilon_o \frac{d\mathbf{E}}{dt} \quad .$$

As with conduction current, the displacement current density is the displacement current per unit area. Because of the time derivative, an increasing electric field makes displacement current in the direction of **E**, while a decreasing electric field makes displacement current in the direction opposite to **E**.

The physical meaning of this displacement current is that a changing electric field makes a changing magnetic field. To find the direction of the magnetic field produced by this effect we use our usual right-hand rule for currents: point your thumb in the direction of the displacement current (the direction of $d\mathbf{E}/dt$), and your fingers will curl around in the direction of the magnetic field. (Note that if the electric field is decreasing in time, then the negative time derivative gives a displacement vector that points opposite to **E**.)

DEAD CONCEPT

30.6 The Ether

Maxwell's ideas made a very pretty theory, but this is not enough. Before accepting his ideas of the displacement current and the ether, we insist on experimental verification. During a period of about 50 years after the publication of Maxwell's theory, there were extensive efforts made to verify that changing electric fields did, in fact, produce magnetic fields in the way Maxwell predicted. Maxwell's theory passed every one of these tests with flying colors. But when experiments were performed to try to study the other properties of the ether, there were difficulties. Every experiment to detect the ether in some way other than by observing the magnetic fields produced by its current was a failure. Finally physicists were forced to the following frustrating conclusion: Maxwell's displacement current term is correct and belongs in the equations of electromagnetism; but the ether idea, on which its discovery was based, is incorrect. This is not very satisfying, but we will soon encounter a similar situation when we study Faraday's law, which is the flip side of the displacement current: a changing magnetic field makes an electric field. There is no simple picture, like that of the displacement current in the ether, to explain this effect. Faraday discovered it experimentally, and we accept it. In the same way, we now accept Maxwell's displacement current term in Ampere's law without believing that it represents any sort of real current in the ether.

ADVANCED CONCEPT

30.7 Maxwell's Equations

With the addition of Ampere's law and Gauss's law for magnetic fields, we are now ready to write down, in incomplete form, all four of Maxwell's equations:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_o} \quad ; \quad \oint \mathbf{E} \cdot d\mathbf{s} = 0 \ .$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad ; \quad \oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I_{enclosed} \ .$$

Note that $I_{enclosed}$ in the last of these equations includes both conduction and displacement currents. After a correction in Chapter 31 these equations will become the complete and justly famous Maxwell's equations.

BASIC CONCEPTS

30.8 Magnets

The reason that matter has magnetic properties is that the fundamental particles are magnetic. The electron, the proton, and the neutron carry dipole magnetic fields with them wherever they go, so it is really no surprise that matter behaves magnetically. What is surprising is that some special forms of matter (iron and other ferromagnetic materials) make such strong magnets. This amazing behavior puzzled natural philosophers for centuries and even the beginning of an explanation had to wait for the discovery of quantum mechanics in this century. With the exception of ferromagnetic materials, however, the magnetic behavior of matter is fairly weak and not so hard to understand.

30.9 Magnetic Intensity and Permeability

When matter is placed in a magnetic field, its atoms respond to the applied field by either adding their own magnetic fields to that of the applied field (ferromagnetic and paramagnetic materials) or by subtracting from it (diamagnetic materials). This extra magnetism produced by the material is described in a rather confusing way, so pay close attention. The first important concept is the *magnetization* $\mathbf{M}$, defined to be the net magnetic dipole moment per unit volume in the material:

$$\mathbf{M} = \frac{\boldsymbol{\mu}}{V}$$

The total magnetic field inside such a material is related to the applied external field and the magnetization by

$$\mathbf{B} = \mathbf{B}_{ext} + \mu_o \mathbf{M} \quad .$$

The *magnetic field strength* $\mathbf{H}$ is a quantity which has the same units as $\mathbf{M}$ but depends only on the strength of the external magnetic field:

$$\mathbf{H} = \frac{\mathbf{B}_{ext}}{\mu_o} = \frac{\mathbf{B}}{\mu_o} - \mathbf{M} \quad .$$

Note that $\mathbf{H}$ has units of magnetization, not units of magnetic field. The *magnetic susceptibility* χ gives the ratio between $\mathbf{M}$ and $\mathbf{H}$:

$$\mathbf{M} = \chi \mathbf{H} \quad .$$

If χ is large, it means that the atoms inside the material respond strongly to the applied magnetic field, while if it is small it means that the magnetism of the atoms in an applied field is not very important. Finally, we introduce the *permeability* μ (not the same as the magnetic moment $\boldsymbol{\mu}$) which is related to the susceptibility by

$$\mu = \mu_o(1 + \chi) \quad .$$

This quantitity gives a simple relation between the total magnetic field in the material and the magnetic field strength $\mathbf{H}$:

$$\mathbf{B} = \mu \mathbf{H} = \frac{\mu}{\mu_o} \mathbf{B}_{ext} \quad .$$

30.10 Atoms as Magnets

Atoms behave as magnets for two reasons. First, the electrons which make up the atom are themselves magnets, with magnetic dipole moments of magnitude one *Bohr magneton* m_B:

$$m_B = 9.27 \times 10^{-24}\ \mathrm{A \cdot m^2} \quad .$$

(Protons and neutrons also have magnetic moments, but they are much smaller in magnitude than m_B.) Second, the atoms are "orbiting" the nucleus, and this orbital motion is often equivalent to circulation of charge, which of course means the electron is like a current loop. Hence, there is the possibility of an orbital dipole moment as well. These orbital dipole moments have magnitudes on the order of a few Bohr magnetons. Different materials respond to applied magnetic fields in different ways because of the various ways the atomic dipole moments respond to the applied field and to the fields of neighboring atoms.

30.11 Ferromagnetism

Ferromagnetic materials are the most magnetically active substances in the world, and so they have very high magnetic susceptibilities, ranging from 1000 up to 100,000. These materials are made of atoms with permanent dipole moments, and when these materials form solids by exchanging electrons to make chemical bonds, something special happens. If the atoms are of the right type and if the bond lengths are right, the electrons discover that they can place the system in a state of lower energy by having neighboring atomic dipole moments aligned with each other. (This sentence probably makes no sense to you, but it is the best we can do. That's just the way quantum mechanics is.) If the entire sample were to be made of aligned dipoles, however, a strong magnetic field would be created, and this would be a state of high energy. So the system compromises. It makes microscopic regions in which billions of dipoles are aligned, satisfying the demands of most of the electron bonds. But the alignment directions of the separate regions are random throughout the sample, making a very weak net magnetic field. These regions are called *magnetic domains*, and their behavior gives ferromagnetic materials their distinctive properties.

For example, if a ferromagnetic material is heated to too high a temperature, it ceases to be ferromagnetic. The reason is that above a certain critical temperature, called the Curie temperature, the thermal motion of the atoms is so violent that the electrons in the bonds are no longer able to keep the dipole moments aligned. When this happens, the ferromagnetic material changes into a paramagnetic material with the usual weak magnetism.

The domains also make permanent magnets possible. If a ferromagnetic sample is placed in a strong magnetic field, the domains can be forced to coalesce into large domains aligned with the external field. When the external field is removed, the electrons in the bonds maintain the alignment and the magnetism remains. This means that ferromagnetic materials can remember their past magnetic history. This property of magnetic memory is called *hysteresis* and lies at the heart of audio tape, video tape, and magnetic disk storage for computers. The recording head of a tape recorder, or the write head of a disk drive, applies a field that magnetizes a small portion of the tape or disk. The magnetism in each portion remains until another magnetic field changes it. When each magnetized section is moved under the playback head of a tape player, or the read head of a disk drive, the moving magnetic field induces small currents which are amplified and turned into either music or data bits. If the domains were unable to remember the field that had been applied to them, none of this would be possible.

Diamagnetism

A *diamagnetic* material is one whose atoms have no permanent dipole moment. When they are placed in a strong magnetic field, Lenz's law acts on the orbiting electrons and causes an atomic dipole moment to appear directed oppositely to the direction of the magnetic field. The effect is very weak, but its effect, roughly, is to cause repulsion where other forms of magnetism give attraction. Because this effect opposes the applied field, the susceptibilities of such materials are negative, and because the effect is weak the magnitudes of the susceptibilites are small, say in the range -10^{-5} to -10^{-4}.

30.12 Paramagnetism

A *paramagnetic* material is one whose atoms do have permanent dipole moments, but the magic of ferromagnetism is not active. If a magnetic field is applied to such a material, the dipole moments try to line up with the magnetic field, but are prevented from becoming perfectly aligned by their random thermal motion. Because the dipoles try to line up with the applied field, the susceptibilities of such materials are positive, but in the absence of the strong ferromagnetic effect, the susceptibilities are rather small, say in the range 10^{-5} to 10^{-3}.

If on the average only a relatively small fraction of the atoms are aligned with the field (say 30% or less), then the magnetization obeys *Curie's law*:

$$M = C\left(\frac{B_{ext}}{T}\right)$$

where C is a constant (different for each different material), where T is the temperature in kelvins, and where B_{ext} is the applied magnetic field. Curie's law says that if B_{ext} is increased, the magnetization increases (the stronger magnetic field aligns more of the dipoles). It also says that if the temperature is increased, the magnetization decreases (the increased thermal agitation helps prevent alignment). Curie's law only works for samples in which only a relatively small fraction of the atoms are aligned, on the average, with the magnetic field. When the aligned fraction becomes larger, Curie's law no longer holds because it predicts that the magnetization just goes up forever with increasing applied magnetic field B_{ext}. But this can't be true because once the dipoles are 100% aligned, further increases in the magnetization are impossible. When this happens we say that the material is saturated, and further increases in B_{ext} or decreases in T will not change the magnetization very much because the atoms are about as aligned as they can get.

When a paramagnetic material is placed in a strong magnetic field, it becomes a magnet, and as long as the strong magnetic field is present, it will attract and repel other magnets in the usual way. But when the strong magnetic field is removed, the net magnetic alignment is lost as the dipoles relax back to their normal random motion.

31 Faraday's Law

BASIC CONCEPTS

31.1 Induced Currents

Michael Faraday discovered that whenever a conductor is moved through a magnetic field, or whenever the magnetic field near a conductor is changed, currents flow in the conductor. This effect is called *electromagnetic induction*. He performed literally hundreds of different experiments, each with its own peculiarities, and made the amazing discovery that all of them could be described by a single simple law, which we call *Faraday's law*. This law gives the magnitude and direction of the emf produced in a conducting loop whenever the loop is moved, or whenever the magnetic field near the loop is changed:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \quad .$$

In this equation Φ_B is the magnetic flux through the loop,

$$\Phi_B = \int \mathbf{B} \cdot d\mathbf{A} \quad .$$

Since our physical interpretation of flux is that it counts the number of field lines that pass through an area, this law says that if the number of field lines through a loop is changed by any means, then an emf will be induced in the loop. The minus sign in Faraday's law indicates the direction of the induced currents, but its use requires so many complicated rules and negative signs that we usually get the direction of current flow from Lenz's law.

31.2 Lenz's Law

The negative sign in Faraday's law indicates the direction of the emf, but it is difficult to properly interpret this sign. A better way to find the direction of the induced emf is to use *Lenz's law*. When we use Lenz's law, we endow the conducting loop with feelings; in fact, we make the loop a violent reactionary. Whenever the number of field lines through the loop is changed, the loop screams "NO", and tries to keep the number of field lines (its flux) the same. It does this by trying to drive current in the direction that produces a magnetic field to oppose the change. If the flux through the loop is decreasing, current flows in the loop in the direction that replaces the lost field lines. If the flux is increasing, the current flow makes magnetic field in the opposite direction to the increasing field.

APPLICATIONS

31.3 Many-Turn Loops

If the conducting loop is actually a coil of N turns of wire all wound in the same direction, then the induced emf's in each turn add to give a larger total emf:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \quad .$$

31.4 Motional Emf

Suppose that a straight segment of wire of length L is moving through a uniform magnetic field, $\mathbf{B}$, with velocity, $\mathbf{v}$. If the wire cuts across the magnetic field lines, Faraday's law predicts that an emf will be produced along the length of the wire, and we call it a *motional emf*. The origin of this emf is simply that the free charges in the wire are moving in a magnetic field, and hence are acted on by the magnetic force. The induced emf in this case is just the $q\mathbf{v} \times \mathbf{B}$ force (discussed in Chapter 29) in disguise. If we define the vector $\mathbf{L}$ to be the length vector that points the long way along the segment of wire, the induced emf along its length is given by the formula

$$\mathcal{E} = \mathbf{v} \times \mathbf{B} \cdot \mathbf{L} \quad .$$

If the field is non-uniform, or if the wire is not straight, then the emf along the wire must be computed by integrating:

$$\mathcal{E} = \int \mathbf{v} \times \mathbf{B} \cdot d\mathbf{L} \quad .$$

One especially important case of motional emf is a loop of wire spinning at constant frequency in a uniform magnetic field. If the loop has area A, if the uniform magnetic field strength is B, and if the angle between the normal to the area and $\mathbf{B}$ is θ, then the flux is given by

$$\Phi = BA\cos\theta \quad .$$

If the loop spins, so that θ changes with time, an emf will be induced in the loop. You might be tempted to find the magnitude of this emf by dividing the total flux change BA by the time it takes it to go from maximum value to zero, but doing it this way gives the wrong answer. To do it right you have to take a derivative. To describe the spinning loop we just let the angle θ change with time according to $\theta = \omega t + \phi$, where ϕ is the angle θ at time $t = 0$. The emf is now easily computed from Faraday's law:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}BA\cos(\omega t + \phi) = \omega BA\sin(\omega t + \phi)$$

Note that this is actually easier than dividing by the time to go from maximum to zero because the magnitude of the oscillating emf is simply given by $\mathcal{E} = \omega BA$. This mathematics describes what happens when the massive coil of an electric generator is forced by a turbine to spin in a strong magnetic field. This spinning emf produces electric current to power our lights, washers, VCRs, refrigerators, air conditioners, etc..

31.5 Induced Electric Fields

When a coil moves past a magnet, we can see that the emf of Faraday's law is just the motional emf produced by the $q\mathbf{v} \times \mathbf{B}$ force. But if the coil sits still and the magnet moves past, Faraday's law still says that there is an induced emf. Since the coil doesn't move, the electrons can't be pushed by the magnetic force, and so the emf must be due to some other effect. This mystery can be solved either by deep thought or by clever experimentation, and the answer is that whenever a magnetic field changes in time, an *induced electric field* is produced. This is a totally new, and very important, effect. Many everyday phenomena and many electronic devices owe their existence to these induced electric fields. AC motors, transformers, audio tape recorders, video recorders, radio, television, the spark plug in a lawnmower, and the interference you see on your television when the vacuum cleaner is running, all work by induced electric fields.

Up to now the only way we knew to make electric fields was to use electric charges. But now that we know that moving magnets also make electric fields, we need to learn what these electric fields look like. The induced electric field lines are usually perpendicular to the changing magnetic field that produces them, and in symmetric situations the electric field lines are circular. For example, suppose that the north end of a bar magnet is headed straight for your nose. The induced electric field lines form rings centered about the axis of the magnet, in the appearance of the rings on an archery target. To determine the direction of the electric field lines, we use Lenz's law in this hypothetical form: Pretend that the circular field lines are conducting wires. The electric field points in the direction that Lenz's law would predict for the induced currents in the wires. In the case of the approaching north pole, the field lines would appear to you to circulate clockwise.

The magnitude of the induced electric field can be calculated, in symmetric situations, from this form of Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \quad .$$

This is another one of those laws (like Gauss's law and Ampere's law) which is always true, but not much help unless there is enough symmetry to make the integral on the left easy to calculate.

ADVANCED CONCEPT

31.6 Maxwell's Equations

The form of Faraday's law given above corrects one of the equations in the set we have been calling Maxwell's equations. Since the electric fields produced by changing magnetic fields circulate, it can no longer be true that $\oint \mathbf{E} \cdot d\mathbf{s} = \mathbf{0}$. The zero in this equation is replaced by Faraday's rate of magnetic flux change, giving Maxwell's equations in their complete and final form. We will discuss further implications of these equations in Chapter 34.

$$\text{Gauss's Law}: \quad \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_o} \quad ; \quad \text{Faraday's Law}: \quad \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_{B\ enclosed}}{dt} \quad .$$

$$\text{Gauss's Law}: \quad \oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad ; \quad \text{Ampere's Law}: \quad \oint \mathbf{B} \cdot d\mathbf{s} = \mu_o I_{enclosed} + \mu_o \epsilon_o \frac{d\Phi_{E\ enclosed}}{dt} \quad .$$

32 Inductance and Circuit Oscillations

BASIC CONCEPTS

32.1 Inductance

When we say that a circuit has inductance, we mean that Faraday's law plays a role in the circuit when the current changes in time. It is important to keep separate the two kinds of inductance, *mutual inductance* and *self inductance*. Although we didn't use the term, we actually studied mutual inductance in Chapter 31. Whenever the magnetic flux of one coil induces an emf in another coil, we have mutual inductance. For example, when the changing magnetic field of overhead power lines induces currents in your car radio, making it impossible to listen to the music, mutual inductance is at work. The mutual inductance between two circuits is defined to be the magnetic flux produced in circuit 1 by the current in circuit 2 divided by the current in circuit 2; or to be the flux produced in circuit 2 by the current in circuit 1 divided by the current in circuit 1:

$$M = \frac{\Phi_1}{I_2} = \frac{\Phi_2}{I_1} \quad .$$

It is not obvious that these two different ratios are the same, but they are; the mutual inductance is truly mutual. Faraday's law then allows us to write the following formula for the induced emf in circuit 2 due to a current change in circuit 1:

$$\mathcal{E}_{21} = -M\frac{dI_1}{dt} \quad .$$

Self inductance is an important effect that we have ignored until now. To understand this idea, imagine a coil turning in a magnetic field. In Chapter 31 we learned that the changing flux through the moving coil induces an emf in the coil. But this is not the whole story. The emf generaged by the spinning drives currents in the coil, and these induced currents also make magnetic flux through the coil, which, in turn, changes the emf of the coil. This is one of the few examples in physics where an object can act on itself, and this effect is called self inductance. Since the purpose of electric circuits is to carry currents, and since currents always make magnetic fields, every circuit has self inductance. It may be possible to make the inductance small, but it can't be made to vanish.

As long as a coil doesn't change its shape, the flux through it is proportional to its own current; this makes it convenient to define the inductance to be

$$L = \frac{N\Phi}{I} \quad ,$$

where Φ is the flux through **one** of the turns of the coil, and where N is the number of turns of wire in the coil. Like the capacitance and the resistance, the inductance of a coil depends only on its geometry and on the materials of which it is made.

32.2 Induced Emf

Since Faraday's law contains the time derivative of the quantity $N\Phi$ found in the definition of the inductance, the induced emf in a coil due to its own changing current can be rewritten

$$\mathcal{E} = -L\frac{dI}{dt} \quad .$$

The minus sign indicates, as usual, that the induced emf opposes change. So if we try to increase the current in a coil, the induced emf points in the direction opposite to the current. And if we try to decrease the current in a coil, the induced emf will point in the direction of the current.

APPLICATIONS

Long Solenoid

The straight uniform magnetic field of the long solenoid makes it possible to find a simple expression for its inductance:

$$L = \mu_o n^2 A\ell = \mu_o \frac{N^2}{\ell} A \quad ,$$

where n is the number of turns per unit length, N is the number of turns, A is the cross-sectional area of the solenoid, and where ℓ is its length. In other inductor geometries, calculating the inductance usually involves complicated integrals.

32.3 Loop Theorem Rule

This effect can be included in our loop-theorem analysis of circuits by using the following rule:

If an inductor is traversed in the direction of the current, then $\Delta V = -Ldi/dt$; if traversed opposite to the direction of the current, $\Delta V = Ldi/dt$.

32.4 Series and Parallel

When connected together into networks, inductors obey the same rules as resistors.

Series: When inductors are connected in series, their equivalent inductance is given by

$$L_{eq} = L_1 + L_2 + L_3 + ...$$

Parallel: When inductors are connected in parallel, their equivalent inductance is given by

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + ...$$

32.5 Inductive Energy

Capacitors store energy, resistors dissipate energy, and energy is important for inductors as well. Inductors are similar to capacitors in that they store energy, and the stored energy is given by the formula

$$U = \frac{1}{2}LI^2 \quad .$$

32.6 Magnetic Energy Density

Just as we thought of the capacitor storing its energy in the electric field between its plates, we may also think of the inductor storing its energy in the magnetic field. The formula for the *magnetic energy density* (energy per unit volume) of a magnetic field is

$$u_B = \frac{B^2}{2\mu_o} \quad .$$

32.7 LR Circuit

Rising Current: If a circuit is switched on, the current cannot instantly rise to the value determined by its resistance because of the opposing emf produced by its inductance. If an emf $\mathcal{E}$, a resistance R, and an inductance L are connected together in series, the loop rules can be used to find the differential equation that determines the way the current rises from zero up to its final value:

$$L\frac{dI}{dt} + RI = \mathcal{E} \quad .$$

The solution of this equation for a circuit with zero current at time $t = 0$ is

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/(L/R)}) \quad .$$

The quantity L/R is called the *inductive time constant*, or the *L-over-R time*, of the circuit; it plays the same role here that the time constant RC played in the resistor-capacitor circuit (Chapter 28). Note that the equation for $I(t)$ is very similar to the equation for $Q(t)$ in a charging capacitor.

Falling Current: It is not so simple to turn a circuit off. If we just open a switch and very quickly stop the current, Faraday's law says that an enormous emf should be produced because of the very short time involved. Very often this enormous emf is large enough to cause the air to break down and spark. This is just Lenz's law at work again, fighting the change in the current, and it is easy to produce this effect. Try turning the vacuum cleaner off by pulling the plug out of the wall, and you will see a flash at the outlet. This flash is caused by Lenz's law in the large inductance of the windings in the vacuum cleaner motor. To avoid a flash, it is necessary to design a special kind of switch that gives the current a conducting path through which to flow while it decreases. If this has been done, then the loop rules applied to a circuit with inductance L and resistance R give

$$L\frac{dI}{dt} + IR = 0$$

whose solution, for initial current I_o, is given by

$$I(t) = I_o e^{-t/(L/R)} \quad .$$

Note that this is similar to the formula for $Q(t)$ during the discharge of a capacitor.

32.8 Plumbing Analogy

It will help us understand the meaning of inductance if we can find its analog in the plumbing system we have been discussing. Suppose a pump is to circulate water in a closed-pipe system. A long time after the pump has been turned on, the flow rate is determined only by the strength of the pump and the resistance of the pipes. But when the pump is first switched on, there is another effect that limits the water flow: inertia. Since the water has mass, it is not possible instantly to set it in motion, so it takes a while for the water to accelerate up to its final speed. Similarly, if the pump is turned off, the inertia of the water will cause its circulation gradually to decrease, rather than to stop immediately. But this is just how inductance affects the flow of current in electrical circuits, so the inductance L is analogous to the mass, or inertia, of the water in the pipes.

BASIC CONCEPTS

32.9 LC Oscillations

If a capacitor is allowed to discharge through a resistor, the charge just exponentially dies away, as discussed in Chapter 28. If a capacitor is allowed to discharge through an inductor, something completely different happens. Since an inductor stores energy, but does not dissipate it, the energy stored in the capacitor must simply be transferred to the inductor during the discharge. The result is an oscillation with the energy repeatedly passed back and forth between the capacitor and the inductor. For convenience, here are the energy formulas again:

$$U_C = \frac{1}{2}CV^2 \qquad ; \qquad U_L = \frac{1}{2}LI^2 \quad .$$

The equation that describes the way the charge on the capacitor varies with time is the following second order differential equation, which may be obtained from the loop theorem in the usual way:

$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = 0 \quad .$$

The solution of this equation corresponding to having charge Q_o on the capacitor at the beginning of the oscillation is

$$Q(t) = Q_o \cos\omega t$$

where the angular frequency of the oscillation is given by

$$\omega = \frac{1}{\sqrt{LC}} \quad .$$

In case you have forgotten how *angular frequency*, *oscillation period*, and *frequency* (in Hz or cycles-per-second) are related, here is a brief review. The period T is the repeat time of the oscillation, i.e., the time between neighboring peaks or troughs of the sinusoidal signal. The frequency f (or sometimes ν) is the number of oscillation cycles that occur per second, or

$$f = \frac{1}{T} \quad .$$

The angular freqency ω is the convenient frequency that makes it possible to describe the oscillation with the simple function $\cos \omega t$. It is connected to f and T by

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The current in the circuit can either be obtained from $I = -dQ/dt$, or from conservation of energy:

$$I(t) = Q_o \omega \sin \omega t \quad .$$

Note that the charge varies as $\cos \omega t$ while the current varies as $\sin \omega t$. This means that the charge and the current are out of phase with each other; we will have more to say about such phase differences in Chapter 33.

32.10 Plumbing Analogy

What about our plumbing analogy, which has helped us understand capacitors, resistors, emfs, and inductors? Could there be a water oscillation similar to these LC oscillations? There is, and it is called a "water hammer". Some of you may have experienced it, but perhaps not, for it is a bit rare. It sometimes happens, when a faucet is turned on, that a loud vibration or banging is heard in the water pipes. If the faucet is fully opened, the banging stops, but if the faucet is open just a little bit, the banging can be quite loud and, in some cases, can even damage the plumbing in the building. The loud sound is caused by the water oscillating back and forth in the pipes, similar to the oscillations in an LC circuit. Because this effect is so rare, it is not of much use in understanding LC oscillations; the mass-spring analogy discussed below does a better job.

32.11 Damped Oscillations

The formulas for the LC oscillator assume that there is no resistance. But, of course, there is almost always resistance, so we need to know how the oscillations change when resistance is included. The simplest way to see what happens is to use an analogy between the electric circuit and an oscillating mass-spring system. The mass is like the inductance, the spring is like the capacitance, and friction is like resistance.

If the friction is very large (imagine putting the oscillating mass in a bucket of honey) and if the spring is not very strong, then the velocity of the mass will be so small that its inertia never becomes important, and the system won't oscillate. If the mass is pulled down, it just slowly returns to its equilibrium position. This is exactly what happens in an RLC circuit if the resistance is very large; the capacitor just slowly discharges through the resistor without any oscillation.

If the friction is very small (imagine letting the mass move in the air) then the mass will oscillate freely, but with a slowly decreasing amplitude. This is also the behavior of the

RLC circuit with small resistance; the charge on the capacitor oscillates many times, but the maximum charge on the capacitor slowly decreases with time. We call such an oscillation a *damped oscillation*.

The differential equation that gives $Q(t)$ for a circuit containing a capacitor C, an inductor L, and a resistor R is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0 \quad .$$

In the case of small enough resistance that oscillations occur, the solution of this equation is

$$Q(t) = Q_o e^{-Rt/2L} \cos \omega_d t$$

where the oscillation frequency, ω_d, is modified from $1/\sqrt{LC}$ by the resistance:

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad .$$

Note that if the resistance becomes too large, the argument of the square root in this formula becomes negative. This indicates that the circuit has changed its behavior from damped oscillations to damping without oscillations, but the new formulas will not be given here. You will encounter them when you study differential equations.

33 Alternating Currents

BASIC CONCEPTS

33.1 Forced Oscillations

The oscillations we have discussed up to now are *free oscillations* in which the system is given some energy, then left alone. For instance, you could pull a child on a swing up to a certain height, then let go and wait for the motion to die away. But this is not the only possibility; we could also repeatedly push the swing at any frequency we like and watch what happens. In this case we say that we have *forced oscillations.* There are now two frequencies in the problem: the *natural frequency* ω_o of the free oscillations, and the *driving frequency* ω of the forced oscillations. This means that you will have to resist the urge to use the formula $\omega = 1/\sqrt{LC}$ whenever you encounter a frequency. If the frequency in question is the driving frequency, there is no formula for it; it is simply set by the design of the driving circuit.

Transients and Steady State: Now, how does the oscillating system respond to this oscillating driver? The equations are too complicated to solve here, so we will use our mechanical analogy again. If the mass-spring system is driven at a frequency other than its natural frequency, a rather complicated motion results. The mass will start to oscillate, then stop, then start up again, and stop again, over and over. The reason for this odd motion is that there are actually two motions occuring simultaneously, one at the natural frequency, ω_o, and one at the driving freqency, ω. If we wait long enough, however, the motion at the natural frequency will die away because of friction, leaving only the motion at the driving frequency. This initial period of complicated motion is called a *transient*, and in this course we will ignore it. We are only interested in the final motion at the driving frequency, which is sometimes called the *steady state* motion. Electric circuits behave in the same way. When you plug an appliance into the wall, an oscillating emf at 60 Hz is applied to it. For a very brief period of time there is a complicated variation of the current with time, but after this transient dies away, the current in the appliance oscillates steadily at 60 Hz.

APPLICATIONS

33.2 Resistor Circuit

If a voltage difference given by

$$V = V_R \sin \omega t$$

is applied across a resistor R, then the resulting current is given by

$$I_R = \frac{V_R}{R} \sin \omega t \quad .$$

Because the voltage and the current both vary as $\sin \omega t$, we say that they are *in phase* with each other.

33.3 Capacitor Circuit

If a voltage difference given by

$$V = V_C \sin \omega t$$

is applied across a capacitor C, then the resulting current is given by

$$I_C = \omega C V_C \cos \omega t \quad .$$

We now try to make this situation similar to the resistor circuit by insisting that the magnitude of the current be given by a formula that looks like Ohm's law:

$$I_C = \frac{V_C}{X_C} \qquad ; \qquad X_C = \frac{1}{\omega C} \quad .$$

The quantity X_C is called the *capacitive reactance*, has units of ohms, and is used just like a resistance to find the magnitude of the current that flows due to an applied voltage.

Note that the applied voltage and the resulting current are not given by the same function of time, meaning that they are *out of phase* with each other. If we start watching this circuit at time $t = 0$, then we see the current at its maximum value and the applied voltage at zero. A quarter of a cycle later the current has dropped to zero and the voltage has risen to its maximum value. We describe this situation by saying that the current is ahead of the voltage, or that it *leads* the voltage. Sometimes we also say that the voltage is behind, or *lags*, the current. It may be helpful to remember the word ICE for this circuit; the C stands for capacitor, and in "ICE" the current, I, is ahead of the voltage, or applied emf, E.

33.4 Inductor Circuit

If a voltage difference given by

$$V = V_L \sin \omega t$$

is applied across an inductor L, then the resulting current is given by

$$I_L = -\frac{V_L}{\omega L} \cos \omega t \quad .$$

We make this situation similar to the resistor circuit by writing

$$I_L = \frac{V_L}{X_L} \qquad ; \qquad X_L = \omega L \quad .$$

The quantity X_L is called the *inductive reactance*, has units of ohms, and is used to find the magnitude of the current that flows due to an applied voltage.

Again, the applied voltage and the resulting current are not given by the same function of time, so they are out of phase. If we start watching this circuit at time $t = 0$, then we see the current at its minimum (negative) value and the applied voltage at zero. A quarter of a cycle later the current has made it up to zero on its way up to its maximum, but the voltage has already risen to its maximum value. We describe this situation by saying that the voltage is ahead of, or leads, the current by 90°. It may be helpful to remember the word ELI for this circuit; the L stands for inductor, and in ELI the voltage, or emf E, comes before the current, I. To remember both the capacitor and the inductor words, remember "ELI the ICEman" who used to help keep things cold until the engineers who developed these circuits put him out of business.

33.5 Series RLC Circuit

If an AC emf given by $\mathcal{E} = V_o \sin(\omega t)$ is used to drive current through a resistor, a capacitor, and an inductor connected in series, then the current through each element must be the same. The voltages across the various elements obey the rules given above, and the sum of these voltages must, by the loop theorem, be equal to the applied emf. This sum must be taken at a particular instant of time, which is complicated because each voltage difference will be at a different part of its cycle. Solving this complicated problem gives the following solution for the current $I(t)$ that flows in the circuit:

$$I(t) = \frac{V_o}{Z} \sin(\omega t - \phi)$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

and where

$$\tan\phi = \frac{X_L - X_C}{R} \quad .$$

Note that the amplitude of I is given by V_o/Z, which looks about like the DC relation $I = V/R$. The quantity Z is called the *impedance*, it has units of ohms, and it plays the same role in AC circuits as resistance does in DC circuits. Unlike the resistance, however, it depends on the driving frequency, so the current that flows in the circuit depends sensitively on the driving frequency. Be careful, however; this formula for the impedance only applies to the **series** RLC circuit. Each different circuit has its own impedance formula.

Resonance: Consider the series RLC circuit discussed above. The formula for the current makes it easy to see how things should be adjusted to get as much current as possible from a given driving emf V_o: simply make Z as small as possible. If the circuit value of R is fixed (as is usually the case) then the only way to get more current is by fiddling with X_L and X_C. And it is clear that the smallest value of Z will be obtained when $X_L = X_C$, which a little algebra shows is equivalent to $\omega = \frac{1}{\sqrt{LC}}$. But this simply says that things should be adjusted so that the driving frequency is equal to the natural frequency of the circuit (without the correction due to resistance). So, if the driving frequency is near the natural frequency, very large currents can result. When a circuit is driven near its natural frequency, we say that it is being driven at *resonance*. And the formula for Z shows that the smaller the resistance of the circuit, the larger the response at resonance will be. This is what makes the radio tuner work. The antenna of the radio picks up radio signals from every station in the area, but only the station whose frequency matches the natural frequency of the tuning circuit will cause large currents to flow in the circuit. These currents, when amplified, are the ones that produce the sound you hear. If the circuit is not properly tuned, then it may pick up two stations equally well, an effect you have probably heard many times.

33.6 Power

The power flow in an AC circuit is quite complicated since the inductor and capacitor can store energy for part of a cycle, then give it back later on. We will ignore these complications and concentrate instead on the resistor which simply receives energy and turns it into heat. Since the current is oscillating, the power received by the resistor also oscillates. We are usually not interested in the details of the ups and downs of the power delivered to the resistor, but only in its average value, which is given by the formula

$$P_R = \frac{1}{2} I_{max}^2 R \quad .$$

Electrical engineers prefer not to remember two different power formulas, one for DC circuits and another for AC circuits, so they have invented a different way of indicating the magnitudes of currents and voltages in AC circuits. The *rms* value of any oscillating quantity is defined to be its maximum value divided by $\sqrt{2}$. For example, the rms values of the applied emf and the current in our circuit are given by

$$V_{rms} = \frac{V_o}{\sqrt{2}} \qquad ; \qquad I_{rms} = \frac{I_{max}}{\sqrt{2}} \quad .$$

When the power formula is written in terms of I_{rms}, it looks just like the DC formula:

$$P_R = I_{rms}^2 R \quad .$$

When AC currents or voltages are simply given without saying whether the values are peak values or rms values, they are rms values. For instance, the standard 110 V supplied to our homes and apartments is an rms voltage; the peak voltage at the wall socket is about 170 V.

Another useful concept related to power in AC circuits is the *power factor.* The power factor is defined by using the relation $\cos\phi = I_{rms}R/V_{rms}$ to write

$$P_R = I_{rms}^2 R \quad = I_{rms} V_{rms} \cos\phi \quad ;$$

the quantity $\cos\phi$ is called the power factor. Note that this factor tells us whether the circuit is mostly resistive ($\phi \approx 0 \Rightarrow \cos\phi \approx 1$), or mostly reactive (capacitive or inductive) so that ($\phi \approx \pm\pi/2 \Rightarrow \cos\phi \approx 0$). Physically this means that if the circuit is mostly resistive all of the power is dissipated as heat or mechanical work but that if the circuit is reactive the energy just bounces around in the system and none is dissipated. Power companies want to deliver useful power to their customers rather than have the power bouncing around through the transmission lines burning out transformers and making capacitors on power poles explode. Hence they monitor their power factor very closely, making sure it is as close to 1 as possible.

33.7 The Transformer

The simplest device involving forced oscillations is the *transformer.* A transformer is simply two coils arranged so that nearly all of the flux made by one of them passes through the other one. This means that if an AC current flows in one of the coils, the resulting AC flux will produce an AC emf in the other one, or to put it another way, they have mutual inductance. The coil which is connected to the driving emf is called the *primary* and the coil in which an emf is induced is called the *secondary.* The primary voltage V_1 and the secondary voltage V_2 are related by the ratio of the number of turns, N_1 and N_2, in the two coils:

$$V_2 = \frac{N_2}{N_1} V_1 \quad .$$

If the transformer is properly made, the power delivered by the primary will be very nearly the same as the power received by the secondary, requiring that the currents in the two coils transform oppositely to the voltages:

$$P_1 = P_2 \qquad \Rightarrow \qquad V_1 I_1 = V_2 I_2 \qquad \Rightarrow \qquad I_2 = \frac{N_1}{N_2} I_1 \quad .$$

The ability to transform voltages at will with a device consisting only of two coils, a piece of iron, and no moving parts, is the main reason that AC current is so widely used.

16 Waves (Chapters 16-18)

BASIC CONCEPTS

16.1 Traveling Waves

Everyone has seen waves in motion, perhaps at the beach, perhaps in a wave pool, or perhaps along a phone cord during a boring conversation. Their most striking feature is their smooth gliding motion. To describe such waves mathematically, we will need to find a way to make functions of space move in this way. Imagine, for example, a long rope suspended along the x-axis. If we quickly push sideways on the rope in the z-direction, we shift the rope away from its equilibrium position. Let this initial displacement z , as a function of x, be given by

$$z = f(x) \quad .$$

If we watch what happens next, it appears that this displacement travels along the rope at some speed v. To describe this moving pulse mathematically, we just change the argument of the function:

$$z(x,t) = f(x - vt) \;\;, \text{ moves to the right} \qquad ; \qquad z(x,t) = f(x + vt) \;\;, \text{ moves to the left} \quad .$$

Note that z is now a function of two variables, x and t; this makes the mathematical description of waves somewhat complicated, and you probably won't be comfortable with it until you have had multivariable calculus.

16.2 Sinusoidal Waves

To describe sinusoidal waves traveling along a rope in the positive x direction we use the formula

$$z(x,t) = A \sin(kx - \omega t) \quad .$$

(If the "-" sign is changed to "+", it will travel in the negative x direction.) The quantity A is the *amplitude* of the wave, and tells us the size of the displacement of the rope. The quantity ω is just the angular frequency of the motion of the rope at any fixed value of x. Just as ω is 2π divided by the period of the wave motion in time, the quantity k is 2π divided by the period of the wave in space:

$$k = \frac{2\pi}{\lambda} \quad ,$$

where λ is called the *wavelength* of the wave, and is the distance from the crest of one wave to the crest of the next. The quantity k is called the *wavenumber*.

16.3 Wave Speed

In many cases, the speed of the waves traveling in a medium is determined completely by the medium. For example, whether you whisper, yell, clap, or stomp, the sound waves you make always travel at about the same speed, 1100 ft/s; you have no control over the wave speed. The wave speed, the frequency, and the wavelength are related by the formula

$$v = \frac{\omega}{k} = \lambda f \quad ,$$

where f is the frequency in cycles per second. This formula says that if the speed of waves in the medium is constant, then high frequency waves will have short wavelengths, and low frequency waves will have long wavelengths.

16.4 Transverse and Longitudinal

Waves come in two basic types: *transverse waves* and *longitudinal waves.* A nice system that has both types of waves is a slinky. If you stretch a slinky across the room, you can excite waves on it by moving it from side to side. The waves travel along the slinky, but the slinky moves sideways. This is the signature of a transverse wave: the wave direction and the vibration direction are perpendicular to each other. Water waves, rope waves, phone cord waves, and electromagnetic waves are of this type. To make longitudinal waves on the slinky, bunch a few coils up near your hand, then release them. You will see a little pulse of "bunchiness" travel along the slinky, reflect from the other end, and return to your hand. This is a longitudinal wave, with the wave direction and the vibration direction parallel to each other. Sound waves are of this type. When you clap your hands, you bunch up some air molecules, and when they spring back they launch a sound pulse through the air.

16.5 Standing Waves

If you have ever been talking on the phone and have idly started to bounce the phone cord, you have probably noticed that at certain bouncing frequencies the cord settles into a large amplitude oscillation. With a little practice you can make oscillation patterns with one, two, three, or even more places along the cord where the amplitude is zero. These oscillations are called *standing waves*, and they share many properties with traveling waves. In fact, one way to think about a standing wave is that it is two traveling waves of the same frequency, but traveling in opposite directions. The relation

$$v = \lambda f = \frac{\omega}{k}$$

holds for standing waves, just as it does for traveling waves. You will need to be careful when you measure the wavelength of a standing wave, however. The wavelength is not the distance between the zero amplitude spots; this distance is only one-half wavelength. If you were to take a snapshot of the phone cord as it vibrated with zero amplitude points only at the ends, your picture would clearly show just a half-period of a sine wave. So just remember that the distance between zero points in a standing wave is one-half wavelength, and everything should work out fine (as in Lab 12, for instance).

34 Maxwell's Equations and Electromagnetic Waves

ADVANCED CONCEPT

34.1 Maxwell's Equations

We now return to the laws of electromagnetism which we completed with the addition of Faraday's induced electric field in Chapter 31. These justly famous equations are called Maxwell's equations:

$$\text{Gauss's Law}: \oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_o} \quad ; \quad \text{Faraday's Law}: \oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_{B\ enclosed}}{dt} \quad .$$

$$\text{Gauss's Law}: \oint \mathbf{B}\cdot d\mathbf{A} = 0 \quad ; \quad \text{Ampere's Law}: \oint \mathbf{B}\cdot d\mathbf{s} = \mu_o I_{enclosed} + \mu_o\epsilon_o\frac{d\Phi_{E\ enclosed}}{dt} \quad .$$

These equations have been tested for well over a century now, and as far as we know, they are correct and complete. Their most spectacular prediction is that changing electric and magnetic fields can make each other by propagating as waves through space. Maxwell's equations predict that these waves should travel at the speed

$$v = \frac{1}{\sqrt{\mu_o\epsilon_o}} = 3\times10^8 \text{ m/s} \quad ,$$

which just happens to be the speed of light. Light, in Maxwell's theory, is an electromagnetic oscillation, as are radio waves, microwaves, infrared waves, X-rays, and gamma rays.

BASIC CONCEPT

34.2 Electromagnetic Waves

Imagine a point charge sitting in the middle of infinite space, its electric field lines radiating straight out to infinity. Suddenly, the charge starts to oscillate, back and forth. What happens to its field lines? And if you are a million kilometers away from the charge, when do you find out that it has begun to oscillate? The answers to these questions are provided by Maxwell's equations, and by the idea of *electromagnetic waves*. Let's answer the second question first. The news that the charge has started to oscillate travels outward from the charge at the speed of light. Hence, you find out about the oscillation after a time $t = (10^9 \text{ m})/(3\times10^8 \text{ m/s}) = 3.33$ s. This is pretty quick, considering the distance, but it is not instantaneous. Now for the answer to the first question: when the charge starts to oscillate, its field lines no longer radiate straight out. Instead, Maxwell's equations predict that the field lines snake out through space in the shape of sinusoidal waves, with the wave crests moving outward at the speed

$$v = \frac{1}{\sqrt{\mu_o\epsilon_o}} \quad .$$

If we substitute the numbers into this formula we get

$$v = c = 3 \times 10^8 \text{m/s} \quad ,$$

the well-known speed of light. When he discovered this connection, Maxwell drew the only reasonable conclusion: light must be an electromagnetic wave of the type produced by oscillating charges. In fact, all electromagnetic waves are produced by oscillating charges. Charges oscillate back and forth in the large antennas of television and radio stations, charges oscillate in the klystrons of radar installations and microwave ovens; charges oscillate in the atoms that radiate in the infrared, visible, ultraviolet, and x-ray parts of the electromagnetic spectrum, and finally, charges oscillate in the nuclei that radiate gamma rays.

Maxwell's equations tell us more about these waves than just their speed. They also tell us that the magnitudes of the electric and magnetic fields in these waves always satisfy the relation

$$B = \frac{E}{c} \quad .$$

This means that even if the electric field amplitude is quite large, the magnetic field will be quite small because c is so large. For this reason we almost always forget about the magnetic field when thinking about what these waves do. For instance, when your car radio antenna picks up a radio station, it is responding only to the electric field of the incoming radio waves.

Maxwell's equations also tell us that these waves are transverse, meaning that both **E** and **B** are perpendicular to the direction the waves are traveling. In addition, **E** and **B** are perpendicular to each other, with their cross product pointing in the traveling direction of the waves:

$$\mathbf{E} \times \mathbf{B} \text{ points in the wave propagation direction} \quad .$$

As an example, let's write down the wave functions for an electromagnetic wave of frequency ω and wavenumber k traveling along the x-axis with an electric field of amplitude E_m pointing in the y-direction.

$$\mathbf{E} = E_m \hat{\mathbf{j}} \sin(kx - \omega t)$$

$$\mathbf{B} = \frac{E_m}{c} \hat{\mathbf{k}} \sin(kx - \omega t) \quad .$$

Note the appearance of x in the argument of the sine function, indicating that the wave travels along the x-axis. Also note that $\mathbf{E} \times \mathbf{B}$ points in the x-direction, as it should. As usual, ω and k are related to each other through the wave speed, as discussed in Chapter 16:

$$\frac{\omega}{k} = \lambda f = c \quad .$$

APPLICATIONS

34.3 Energy and Power

Anyone who has touched a hot steering wheel in the late afternoon should realize that electromagnetic waves carry energy. The most convenient way to describe this energy is to write down a formula for the energy per unit volume, u_{EM}, in the space occupied by an electromagnetic wave. Fortunately, this is not a completely new formula but just a combination of two old ones, namely the formulas for the energy densites of electric and magnetic fields:

$$u_{EM} = u_E + u_B = \frac{\epsilon_o E^2}{2} + \frac{B^2}{2\mu_o} \quad .$$

It is easy to show that these two contributions to the energy of the wave are exactly equal. This formula can be used to derive a formula for the *Poynting vector* **S** of an electromagnetic wave. This vector quantity is defined to be the power per unit area in the electromagnetic wave, and points in the propagation direction of the wave. It is given by the formula

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_o} \quad ,$$

and has units of watts/meter^2. Because **E** and **B** oscillate rapidly and are in phase with each other, **S** oscillates rapidly between zero and its maximum value, just like the power in a resistor in an AC circuit. In Chapter 33 we took a time average and re-expressed the power formula in terms of rms values, and it is also convenient to do so here. The time average of the magnitude of the Poynting vector is usually called the *intensity* I of the electromagnetic wave and is given by

$$I = < S > = \frac{1}{2} \frac{E_m B_m}{\mu_o} = \frac{E_{rms} B_{rms}}{\mu_o} = c\epsilon_o E_{rms}^2 \quad ,$$

where we used $B = E/c$ to get the final formula. As an example, consider the radiation from the sun. The intensity of solar radiation just outside the earth's atmosphere is about 1.4 kW/m^2. But down here on the ground in Provo the intensity is considerably less, say 300 - 350 W/m^2 on a sunny day, depending on the humidity. If you lie down on a blanket to get a tan when $I \approx 350$ W/m^2, the power incident on your body is simply the product of I with your area. Assuming you have an area of 0.5 m^2, about 175 W of radiant power shines on your body.

34.4 Radiation Pressure

You may have read science fiction stories in which the hero, fed up with an increasingly polluted and disgusting earth, hooks his life support pod to a gigantic silvery plastic sail out beyond the orbit of Mars, and sails off to the far reaches of the solar system, powered by the pressure of sunlight. This is probably nuts, but sunlight would exert some force on such a sail. Light does carry momentum as well as energy; it's just that the momentum is very small. There are two ways to write the formulas for this effect, and both will be given here.

If you know the intensity, $I = < S >$, of the radiation, then it is most convenient to use a formula for the pressure, P, (force per unit area) exerted by the radiation. The radiation pressure depends on whether the radiation is absorbed or reflected. If it is reflected, the object which reflects it must recoil with enough momentum to stop the incoming wave, and then send it back out again, while if it is absorbed, the object must only stop it. Hence, the reflection formulas give answers that are twice the size of those given by the absorption formulas. The radiation pressure formulas are

$$\text{Pressure}: \quad P = \frac{I}{c} \ \text{(absorption)} \qquad p = \frac{2I}{c} \ \text{(reflection)} \quad .$$

If you know the total amount of energy U in a pulse of radiation, as you might for a laser pulse, then it is most convenient to use formulas for the amount of momentum p imparted to an object that either absorbs or reflects the pulse:

$$\text{Momentum}: \quad p = \frac{U}{c} \ \ \text{(absorption)} \qquad P = \frac{2U}{c} \ \ \text{(reflection)} \ .$$

Because of the factor of c in the denominators of these pressure and momentum formulas, these effects are usually quite small.

34.5 Polarization

We say that an electromagnetic wave is *polarized* if its electric field oscillates up and down along a single axis. For example, a vertical radio broadcasting antenna always makes electric fields that point either up or down, never sideways. Hence, the waves it makes are vertically polarized. The light from a light bulb is unpolarized, however, because the atoms that radiate the light are unorganized. When their light reaches your eyes the electric field might be vertical for a while, then rotate around to horizontal, then back up to vertical, in random fashion. To polarize light we use a material, like Polaroid, that absorbs light polarized in one direction, while leaving the other direction unaffected. Polaroid sunglasses are made to absorb horizontally polarized light because the glare from electrically insulating surfaces (plastic, paint, glass, water, etc.) is mostly horizontally polarized. Try tilting your head while wearing Polaroid sunglasses to see this glare absorbing effect. You might also try looking through such sunglasses at the sky on a sunny day; you should discover that the sky is also polarized. You will learn more about such matters (even why the sky is blue) in Physics 221 and in Optics, Physics 471.

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