Tests for a Conservative Force

There are five equivalent tests for a conservative force, *i.e.*, if a force, \( \mathbf{F} \), passes any one of these tests, it passes all five and is conservative.

(1) The line integral of a conservative force depends upon the end points only, not upon the path taken. Mathematically stated,
\[
\int_{r_1}^{r_2} \mathbf{F} \cdot dr = f(r_A, r_B).
\]

(2) The line integral of the force about any closed path is zero, *i.e.*, the work done by any conservative force on a particle which moves about a closed path is zero. Mathematically stated,
\[
\oint \mathbf{F} \cdot dr = 0.
\]

(3) The force, \( \mathbf{F} \), is purely a function of position (and therefore does not depend upon other variables such as speed, direction of motion, time, etc.). That is, \( \mathbf{F} = \mathbf{F}(\mathbf{r}) \).

(4) There exists a potential function \( U(\mathbf{r}) \), for the force \( \mathbf{F}(\mathbf{r}) \), or mathematically stated, there exists a function of position, \( U(\mathbf{r}) \), such that
\[
\mathbf{F}(\mathbf{r}) = \nabla U(\mathbf{r}) = -\frac{\partial U}{\partial x} \hat{i} - \frac{\partial U}{\partial y} \hat{j} - \frac{\partial U}{\partial z} \hat{k}.
\]
In one-dimensional context this would be stated as
\[
F(x) = -\frac{dU}{dx}.
\]

By integration these statements are equivalent to
\[
U(\mathbf{r}) - U(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r},\quad \text{or } U(x) - U(x_0) = -\int_{x_0}^{x} F \, dx.
\]

(5) The curl of \( \mathbf{F} \) is zero. Mathematically stated,
\[
\nabla \times \mathbf{F} = \begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{bmatrix} = 0.
\]