**Projectile Motion**

The relationships given here are idealized and therefore do not yield highly accurate results in many real world situations. That is because the following simplifying assumptions have been made, none of which is completely true:

1. Air resistance is negligible, (1') also air buoyancy is negligible,
2. $g$, the acceleration of gravity, is constant in magnitude* over the entire particle trajectory.
3. $g$, the acceleration of gravity, is constant in direction* over the entire particle trajectory.
4. The earth is not rotating.

For small trajectories, i.e., small in comparison with the size of the earth, most of the error in most calculated trajectories arises from assumption (1) above.

The projectile is launched at ground level, over flat terrain with elevation angle $\theta$ and initial speed $v_0$. (For simplicity we take the plane of motion to be the $x$-$y$ plane and we orient the $x$-axis horizontally and the $y$-axis vertically.) The acceleration is $a = (0, -g)$, the initial velocity of the projectile is $v_0 = (v_0 \cos \theta, v_0 \sin \theta)$ and the initial position of the projectile is taken as the coordinate origin, i.e., $r_0 = (0, 0)$. Hence the velocity of the projectile is $v = (v_0 \cos \theta, v_0 \sin \theta - gt)$ and the position of the projectile is given by $r = [(v_0 \cos \theta)t, (v_0 \sin \theta)t - \frac{1}{2}gt^2]$. Since $x = (v_0 \cos \theta)t$ and $y = (v_0 \sin \theta)t - \frac{1}{2}gt^2$, we can eliminate $t$ from these two equations to obtain the path of the trajectory in the $xy$-plane:

$$y = \left(\tan \theta\right)x - \frac{gx^2}{2v_0^2 \cos^2 \theta}.$$ This is the equation of a parabola. The range, $R$, of the projectile can be obtained by setting $y = 0$ and solving the last equation for $x$: $R = x(y = 0) = \frac{v_0^2 \sin 2\theta}{g}$. (Note that the second root of this quadratic equation yields $x(y = 0) = 0$, which is not the range, but the initial position of the projectile.) The maximum height achieved by the projectile, $y_{\text{max}}$, is obtained by noting that at the top of the trajectory $v_y = 0$, $t = (v_0 \sin \theta)/g$ and therefore $y_{\text{max}} = \frac{(v_0 \sin \theta)^2}{2g}$. Note that we can obtain $v_y$ as a function of $y$ by eliminating $t$ in the relationships for $y(t)$ and $v_y(t)$. The result is $v_y(y) = \pm[v_0^2 \sin^2 \theta - 2gy]^{1/2}$. The positive root is chosen to the left of $y = y_{\text{max}}$; the negative root is chosen to the right. Note that $v_y(y_{\text{max}}) = 0$. Also note that $v = (v_x^2 + v_y^2)^{1/2} = [(v_0 \cos \theta)^2 + (v_0 \sin \theta)^2 - 2gy]^{1/2} = [v_0^2 - 2gy]^{1/2}$. No matter whether the projectile is ascending or descending, at common values of $y$ the speed of the projectile (but not the velocity) is the same.
If the earth were a perfectly spherically symmetric body of radius $R$, then the magnitude of $g$, at its surface, would be $g_0 = \frac{GM}{R^2}$, where $G = 6.67428 \times 10^{-11}$ N·m²/kg² is the universal gravitational constant, $M$ is the mass of the earth and $R$ is the radius of the earth. The direction of $g$ would be towards the center of the earth. At an altitude $y$ above the surface of such an earth the value $g(y) = g_0 \left(1 + \frac{y^2}{R^2}\right)$. (We can treat $g$ as a constant only if $r/R < 1$.) Even this expression is an oversimplification since the earth is not a perfect sphere and since its structure is only approximately symmetric. Clearly the direction of $g$ also changes along any trajectory which is not perfectly radial. In fact the direction of $g$ (the approximate direction of the center of the earth) changes by roughly 1” for every 100 horizontal feet traversed by the trajectory.

If we take into account the slight variations in both $g$ and the direction of the gravitational acceleration along the trajectory (assumptions #2 and #3 above), and if we are able to use assumptions #1 and #4 above, then the trajectory is slightly altered to become an ellipse rather than a parabola. This indeed is the path type followed by an orbiting satellite, above the earth's atmosphere where air viscosity and buoyancy are virtually zero. For the very small segment of such an orbit, traversed by a freely moving object near the earth's surface, a parabolic segment and an elliptical segment are virtually indistinguishable.