Gravitational Potential Energy

Recall that when all conservative forces (such as gravity) have an associated potential energy. Within the limited confines of near-earth space the force on a body of mass $m$ is given by $\mathbf{F} = -mg \mathbf{j}$ and the associated potential energy is given by $U(y) = U(y_0) - \int_{y_0}^{y} mg \cdot \mathbf{j} \cdot dy = U_0 + mg(y - y_0)$, where $U_0 \equiv U(y_0)$ is an arbitrary constant. Generally it is convenient to set $U_0 = U(y_0) = mgy_0$ so that $U(y) = mgy$.

More generally, when the spatial volume is not so confined, the gravitation force due to a two-body interaction, with body masses of $M$ and $m$, is mutually attractive and has magnitude $F = \frac{GMm}{r^2}$, where $r$ is the center-to-center distance between the two bodies and $G$ is the universal gravitational constant. Hence the force on the body of mass $m$ is given by $F = \frac{GMm}{r^2} \mathbf{r}$, where $\mathbf{r}$ is a unit vector oriented so as to point from the center of body $M$ towards the center of body $m$.

To obtain the potential energy associated with this more general expression we again integrate:

$$U(r) = U(r_0) - \int_{r_0}^{r} \mathbf{F} \cdot d\mathbf{r} =$$

$$U(r_0) - \int_{r_0}^{r} \frac{GMm}{r^2} \mathbf{r} \cdot d\mathbf{r} =$$

$$U(r_0) - \left[ \frac{GMm}{r} \right]_{r_0}^{r} = U_0 + GMm \left( \frac{1}{r_0} - \frac{1}{r} \right).$$

Again we have the options of setting our reference position $r_0$ and our potential energy at that position, $U_0 \equiv U(r_0)$, as anything we want them to be. For convenience we choose $r_0 = \infty$ and $U(r_0) = U(\infty) = 0$. This choice yields the simple result $U(r) = -\frac{GMm}{r}$. This choice also implies that the total energy of a two-body system, $E = K + U$, in which the two bodies are infinitely far apart and stationary, is $E = 0$.

**Velocity of Escape**

Since the final description above of a two-body system will be the ultimate fate of two bodies separating at their mutual escape velocity, and since the energy of such a system is conserved, we can readily determine the velocity of escape of a body of mass $m$ from a body of mass $M$ having a center-to-center separation of $r$ by setting $E = 0$ and solving for $v \equiv v_{esc}$:

$$0 = E = K + U(r) = \frac{1}{2}mv_{esc}^2 - \frac{GMm}{r}. \quad \text{(It has been assumed in this derivation that } M \gg m.)$$
We note that for the earth \( v_{esc} = \sqrt{\frac{2(6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2})(5.97 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}}} = 1.12 \times 10^4 \text{ m/s} = 11.2 \text{ km/s} \). For the sun at a distance of 1 AU \( v_{esc} = \sqrt{\frac{2(6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2})(1.989 \times 10^{30} \text{ kg})}{1.496 \times 10^{11} \text{ m}}} = 4.213 \times 10^4 \text{ m/s} = 42.13 \text{ km/s} \).

### The Energies of Various Orbital Types

By choosing to set \( U(\infty) = 0 \) we have established that a body which marginally escapes so that \( K(\infty) = 0 \), also has \( E(\infty) = U(\infty) + K(\infty) = 0 \). This implies that a body in a bound or elliptical orbit, since it lacks the energy necessary to escape, has \( E < 0 \) while a body capable of escape and in a hyperbolic orbit is everywhere along its trajectory traveling at a speed exceeding the local velocity of escape and has \( E > 0 \). A body with a parabolic orbit is a special case on the boundary between bound and unbound orbits and everywhere has \( v = v_{esc} \) and \( E = 0 \). Hence if we know the speed of a body at a distance \( r \) from the sun where \( E = K + U = \frac{1}{2}mv^2 - GMm/r \) and \( v_{esc} = (2GM/r)^{\frac{1}{2}} \) we can determine its orbital type about the sun. In particular if \( v^* < v_{esc} = (2GM/r)^{\frac{1}{2}} \) then \( E < 0 \) and the body is in an elliptical orbit, if \( v^* = v_{esc} = (2GM/r)^{\frac{1}{2}} \) then \( E = 0 \) and the body is in a parabolic orbit, and if \( v^* > v_{esc} = (2GM/r)^{\frac{1}{2}} \), then \( E > 0 \) and the body is in a hyperbolic orbit. Also note that if \( E < 0 \), the orbit is bound and if \( E \geq 0 \), the orbit is unbound.

*Note that \( v^* \) is the speed with respect to the sun, not the earth.*