The Angular Momentum of a System about a General Point
(not necessarily the mass center)

In the figure at the left we have placed the origin of our coordinate system at point O about which we plan to calculate the angular momentum of a system of many particles. For purposes of illustration, and for simplicity, we have indicated the position of the \(i\)th particle of that system as well as the mass center the system. In the figure \(R\) represents the position vector of the mass center in the elected coordinate system. The position of the \(i\)th particle is represented by the vector \(r_i\); \(p_i\) is the linear momentum of that particle, and \(p_i\) is the position of that particle with respect to the mass center. Clearly

\[
r_i = R + \rho_i.
\]

The angular momentum of the \(i\)th particle is \(L_i = r_i \times p_i\). Hence

\[
L_{\text{sys}} = \sum L_i = \sum r_i \times p_i = \sum (R + \rho_i) \times p_i
\]

\[
= \sum (R \times p_i) + \sum (\rho_i \times p_i) = R \times \sum p_i + \sum (\rho_i \times p_i).
\]

In this final expression \(\sum p_i = p_{\text{sys}}\) and \(\sum (\rho_i \times p_i) = L_{\text{CM}}\), the system's angular momentum about the center of mass. Thus, finally we can conclude that

\[
L_{\text{sys}} = R \times p_{\text{sys}} + L_{\text{CM}}.
\]

For an extended object, we can follow a parallel argument using infinitesimal mass elements instead of particles. The result can be stated: the angular momentum of the system (body) about an arbitrary point (O) equals the angular momentum that would result if the mass of the entire system (body) were concentrated at the mass center, moving with linear momentum \(p = p_{\text{sys}}\) plus the angular momentum of the system (body) about the mass center, \(L_{\text{CM}}\).