Precession of a Top

The figure below is a partial free-body diagram for a spinning top. To keep the diagram relatively uncluttered the top image itself has been suppressed, but it spins on a horizontal surface with a contact point at O. CM is the top's mass center and its rotational axis has been drawn making an angle $\theta$ to the vertical (z-axis). The angular momentum vector $\vec{L}$, the angular velocity vector, $\vec{\omega}$, and the gravitational force vector, $-mgk$ are shown. The $xy$-plane is horizontal and passes through the mass center; the $xz$-plane is the plane of the rotational axis and the vertical precession axis (vertical line passing through the point O).

\[
\vec{\omega}_0 (excited by gravity) = mgd \sin \theta \uparrow
\]

But $\vec{\omega}_0 = \frac{d\vec{L}}{dt} \Rightarrow d\vec{L} = \vec{\omega}_0 \, dt$

Originally $\vec{L} = L \sin \theta \hat{i} + L \cos \theta \hat{k}$

The horizontal component of $\vec{L}$, $L \sin \theta$, changes in direction with the addition of $d\vec{L}$ by $d\phi = \frac{d\vec{L}}{L \sin \theta}$.

The rate of change of direction of $\vec{L}$ is therefore

\[
\Omega = \frac{d\phi}{dt} = \frac{d\vec{L}/dt}{L \sin \theta}
\]

\[
\Omega = \frac{\vec{\omega}_0}{L \sin \theta} = \frac{mgd \sin \theta}{L \sin \theta} = \frac{mgd}{L} = \frac{mgd}{I_W}
\]

The direction of $\vec{L}$ is said to precess about the vertical at angular frequency $\Omega$. 