Derivation of the Angular Form of Newton’s 2nd Law

We begin with Newton’s 2nd Law. A body of mass $m$ which is at position $r$ with respect to an arbitrary point $P$ experiences a net force $F$. By Newton’s 2nd Law, that produces an acceleration given by $F = ma$. Multiplying both sides of this equation by $r$ in a vector cross-product multiplication yields

$$r \times F = r \times ma.$$ 

Therefore $|r \times F| = |r \times ma| = mra \sin \theta$. But, since $F$ and $a$ must be parallel, $a \sin \theta = \alpha$, the tangential component of $a$ with respect to point $P$, and therefore $|r \times F| = mra = mra \alpha$. Finally we write $|r \times F| = mra = mr^2(a_i/r)$, and since $(a_i/r) = \alpha$, the angular acceleration of $m$ about $P$, we can write $|r \times F| = mr^2(\alpha_i/r) = mr^2 \alpha = l \alpha$, where $l = mr^2$ is the rotational inertial of $m$ about $P$. To conclude we note that, since the direction of $r \times F$ (perpendicular to and out of the plane of the figure) is the same as that of $\alpha$,

$$r \times F = l \alpha,$$

which is the rotational form of Newton’s 2nd law. This expression holds for extended objects just as it does for point masses. (We note that changing the position of point $P$ changes the values of both $r$ and $\alpha$, but does not affect the validity of the equality $r \times F = l \alpha$.)

We call the quantity $r \times F$ “torque.” It is customarily represented by the symbol $\tau$, the Greek letter tau, i.e.,

$$\tau \equiv r \times F = l \alpha.$$

This relationship holds true for extended bodies as well as for particles. Hence, if a body is exhibiting no angular acceleration, i.e., if $\alpha = 0$, then we can conclude that $\tau_{\text{net}} = 0$ for that body, for the torque about any point $P$.

Units of Torque: We note that the SI unit of torque is $r[m] \times F[N] = \tau[N\cdot m]$. The dimensions of torque, $ML^2T^{-2}$, are the same as the dimensions of work or energy, but we do not use the units of work or energy, e.g., the Joule, erg, calorie, kilowatt-hour, etc., to represent torque since we prefer to use a unit that reminds us that torque is the result of a force [N] acting with a certain “lever arm” [m]. (The one notable exception to this rule is the foot-pound, an explicit product of a force and a distance which is also a commonly used British energy unit.)